Analysis of 3D Shapes (IN2238)

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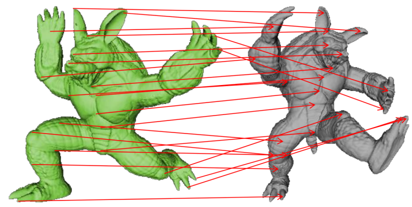
14. Stiffness matrix

Recap

Shape matching

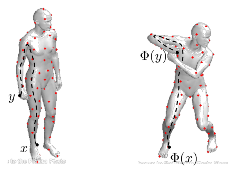


Our goal is to assign each point on the source shape a corresponding point on the target shape. Although a diffeomorphic (bijective and diff'able in both directions) mapping is desired, most of the approaches we discuss will not even guarantee injective mappings (remember nearest neighbors from ICP). Eventually we deal with discretized shapes, mostly triangular meshes. The correspondence will then be a mapping between the vertices.

Isometries

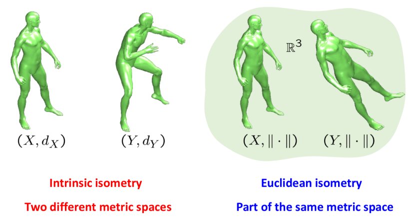
A mapping $\Phi : \mathcal{M} \to \mathcal{N}$ between two shapes (manifolds) is an isometry if

$$d_{\mathcal{M}}(x, y) = d_{\mathcal{N}}(\Phi(x), \Phi(y)) \quad \text{for all points } x, y \in \mathcal{M}.$$



If such a mapping exists $\mathcal{M}$ and $\mathcal{N}$ are called isometric. Many shape matching approaches assume that the shapes to be matched are (nearly) isometric. The task then becomes to find the (almost-)isometry Φ.

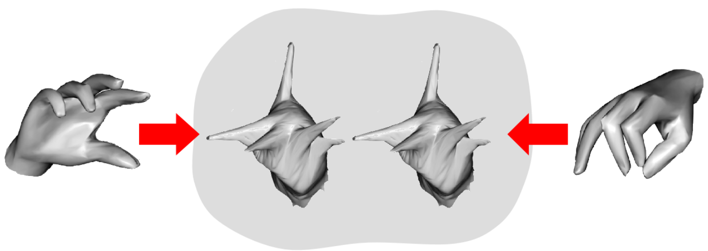
Euclidean isometry



Last week we have seen two methods (ICP and PCA) that can be used to find a correspondence between shapes that are isometric with respect to the Euclidean metric (rigid alignment).
Today we discuss a way to transform the more difficult problem of intrinsic isometries into a rigid alignment problem.

Canonical forms

The main idea is to transform the two shapes to be matched into **canonical forms** such that the two canonical forms are isometric with respect to the euclidean metric.



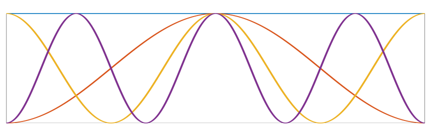
Dirichlet energy

As an alternative to MDS a popular approach to embed a shape $\mathcal{M}$ into a Euclidean space is by finding functions $\varphi_i : \mathcal{M} \to \mathbb{R}$ that are orthonormal *i.e.*$\langle \varphi_i, \varphi_j \rangle_{L^2(\mathcal{M})} = \delta_{ij}$ and minimize the **Dirichlet energy**

$$E_D(\varphi_i) = \int_{\mathcal{M}} \langle \nabla \varphi_i, \nabla \varphi_i \rangle dp = \int_{\mathcal{M}} \|\nabla \varphi_i\|^2 \, dp.$$

The Dirichlet energy measures how *variable* a function is. Let $\mathcal{M} = (-\pi, \pi)$, then

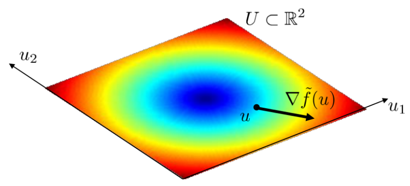
$$E_D(\cos(kx)) = \int_{-\pi}^{\pi} \|\nabla \cos(kx)\|^2 \, dx = k^2 \int_{-\pi}^{\pi} \sin^2(kx) dx = k^2 \left[\frac{x}{2} - \frac{\sin(2kx)}{4k} \right]_{-\pi}^{\pi} = \pi k^2$$



Gradient

We have yet not defined what the gradient of a function $f:\mathcal{M}\to\mathbb{R}$ is. We do know gradients of functions defined on Euclidean domains. For a function $\tilde{f}:\mathbb{R}^2\to\mathbb{R}$ the gradient is given by

$$\nabla\tilde{f}=\begin{pmatrix}\frac{\partial\tilde{f}}{\partial u_1}\\ \frac{\partial\tilde{f}}{\partial u_1}\end{pmatrix}$$

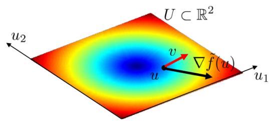


Geometric meaning of the gradient

Geometric meaning of the gradient

- the vector that points in the **direction of steepest increase** of $\tilde{f}$
- its length measures the strength of increase
- relationship with the differential of $\tilde{f}$:

$$\begin{aligned}d\tilde{f}(u)(\vec{v})&=\lim_{h\to 0}\frac{\tilde{f}(u+h\vec{v})-\tilde{f}(u)}{h}\\&=\frac{d}{dh}\tilde{f}(u+h\vec{v})|_{h=0}\\&=\langle\nabla\tilde{f}(u),\vec{v}\rangle\end{aligned}$$



Gradient on manifold

Let $f:\mathcal{M}\to\mathbb{R}$ be a differentiable function. The **gradient** $\nabla f(p)$ at $p\in\mathcal{M}$ is the unique element of $T_p\mathcal{M}$ such that

$$\langle\nabla f(p),\vec{v}\rangle=df(p)[v]$$

In local coordinates

Let $p=x(u)$. Given $\nabla\tilde{f}(u)\in\mathbb{R}^2$ and the first fundamental form $g(u)\in\mathbb{R}^{2\times 2}$, the coefficients $\alpha\in\mathbb{R}^2$ (local coordinates) of $\nabla f=Dx\cdot\alpha\in T_p\mathcal{M}$ are given by

$$\alpha=g^{-1}(u)\nabla\tilde{f}(u)$$

. Let $\beta\in\mathbb{R}^2$ be the coefficients of $\vec{v}\in T_p\mathcal{M}$. Then

$$df(p)[\vec{v}]=\langle\nabla\tilde{f}(u),\beta\rangle=\langle\alpha,\beta\rangle_{g(u)}=\langle\nabla f,\vec{v}\rangle$$

Notice that this in general is a different vector then $\nabla\tilde{f}(u)$!

Stiffness matrix

Stiffness matrix

$$E_D(\varphi_i)=\int_{\mathcal{M}}\langle\nabla\varphi_i,\nabla\varphi_i\rangle dp=\int_{\mathcal{M}}\|\nabla\varphi_i\|^2\,dp.$$

Given two arbitrary functions $f,g:\mathcal{M}\to\mathbb{R}$ we can also consider

$$\int_{\mathcal{M}}\langle\nabla f,\nabla g\rangle dp$$

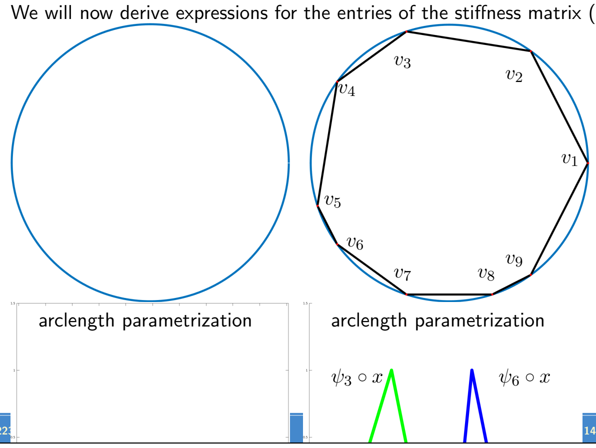
Let $f(x)=\sum_{i=1}^{V}\mathbf{f}_i\psi_i(p)$ and $g=\sum_{j=1}^{V}\mathbf{g}_j\psi_j(p)$ now be PL functions defined on a triangular mesh (ψ_i being hat functions).

$$\int_{\mathcal{M}}\langle\nabla f,\nabla g\rangle dp=\sum_i\sum_j\mathbf{f}_i\mathbf{g}_j\underbrace{\int_{\mathcal{M}}\langle\nabla\psi_i,\nabla\psi_j\rangle dp}_{\mathbf{S}_{ij}}=\mathbf{f}^T\mathbf{S}\mathbf{g}$$

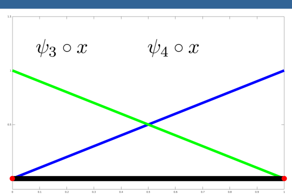
The symmetric (but not pos. definit!) matrix $\mathbf{S}$ is called **stiffness matrix**.

Stiffness matrix - 2D, 1/3

We will now derive expressions for the entries of the stiffness matrix (first in 2D).



Stiffness matrix - 2D, 2/3



parametrization of $e_{ij}=(v_i,v_j)$ from reference interval $[0,1]$

$x(t)=(1-t)v_i+tv_j$

$g(t)=\|e_{ij}\|^2$

$$\begin{aligned}\int_{e_{ij}}\langle\nabla\psi_i(p),\nabla\psi_j(p)\rangle dp&=\int_0^1\langle g^{-1}\nabla\psi_i(x(t)),g^{-1}\nabla\psi_j(x(t))\rangle_g\sqrt{g}dt\\&=\int_0^1\frac{1}{\|e_{ij}\|^2}\langle-1,1\rangle\|e_{ij}\|\,dt\\&=-\frac{1}{\|e_{ij}\|}\end{aligned}$$

Stiffness matrix - 2D, 3/3

$$\mathbf{S}_{ij}=\int_{\mathcal{M}}\langle\nabla\psi_i(p),\nabla\psi_j(p)\rangle dp=\begin{cases}0 & \text{if } (v_i,v_j)\notin\mathcal{E}\\ -\frac{1}{\|e_{ij}\|} & \text{if } (v_i,v_j)\in\mathcal{E}\\ \sum_{k\in\mathcal{N}(i)}\frac{1}{\|e_{ik}\|} & \text{if } i=j\end{cases}$$

In the special case where all the edges have the same length $e_{ij}=s$, the stiffness matrix is given by:

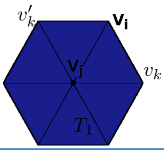
$$\mathbf{S}=\frac{1}{s}\begin{pmatrix}2 & -1 & 0 & & & & -1\\ -1 & 2 & -1 & 0 & & & 0\\ 0 & -1 & 2 & -1 & 0 & & 0\\ \vdots & & & & & & \vdots\\ 0 & & & & & \ddots & -1\\ -1 & 0 & \dots & & & -1 & 2\end{pmatrix}$$

Stiffness matrix - 3D, 1/4

We want to derive $\mathbf{S}_{ij} = \int_{\mathcal{M}} \langle \nabla\psi_i(p), \nabla\psi_j(p)\rangle dp$ for triangular meshes. Due to the localized support of the basis functions we observe:

$$\mathbf{S}_{ij} = \begin{cases} 0 & \text{if } (v_i, v_j) \notin \mathcal{E} \\ \int_{T_{ijk}} \langle \nabla\psi_i(p), \nabla\psi_j(p)\rangle dp + \int_{T_{ijk'}} \langle \nabla\psi_i(p), \nabla\psi_j(p)\rangle dp & \text{if } (v_i, v_j) \in \mathcal{E} \\ \sum_{i \in T} \int_T \|\nabla\psi_i(p)\|^2 \, dp & \text{if } i = j \end{cases}$$

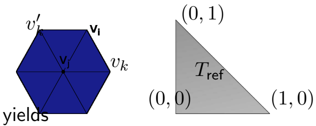
where k and k' are such that $(v_k, v_i, v_j), (v'_k, v_i, v_j) \in \mathcal{F}$ and the sum in the third case is over all triangles T having v_i as a vertex.



Stiffness matrix - 3D, 2/4

Next we derive $\int_{T_{ijk}} \langle \nabla\psi_i(p), \nabla\psi_j(p)\rangle dp$.
Lets first recap the parametrization of the triangle T_{ijk} via

$$x(u) = v_k + u_1 \underbrace{(v_i - v_k)}_{e_1} + u_2 \underbrace{(v_j - v_k)}_{e_2}$$



For the first fundamental form and it inverse this yields

$$g(u) = \begin{pmatrix} \|e_1\|^2 & \langle e_1, e_2\rangle \\ \langle e_1, e_2\rangle & \|e_2\|^2 \end{pmatrix} \qquad g^{-1}(u) = \frac{1}{\det g} \begin{pmatrix} \|e_2\|^2 & -\langle e_1, e_2\rangle \\ -\langle e_1, e_2\rangle & \|e_1\|^2 \end{pmatrix}$$

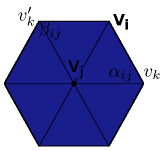
Moreover

$$\tilde{\varphi}_i(u) = \varphi_i(x(u)) = u_1, \nabla\tilde{\varphi}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \tilde{\varphi}_j(u) = \varphi_j(x(u)) = u_2, \nabla\tilde{\varphi}_j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Stiffness matrix - 3D, 3/4

Putting all the pieces together we derive

$$\begin{aligned} \int_{T_{ijk}} \langle \nabla\psi_i(p), \nabla\psi_j(p)\rangle dp &= \int_{T_{ref}} \langle g^{-1}\nabla\tilde{\psi}_i(u), g^{-1}\nabla\tilde{\psi}_j(u)\rangle_g \sqrt{\det g}\, du \\ &= \int_{T_{ref}} \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \|e_2\|^2 & -\langle e_1, e_2\rangle \\ -\langle e_1, e_2\rangle & \|e_1\|^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{\det g}} du \\ &= -\frac{1}{2} \frac{\langle e_1, e_2\rangle}{\sqrt{\det g}} = -\frac{1}{2} \frac{\|e_1\| \, \|e_2\| \cos(\alpha_{ij})}{\|e_1\| \, \|e_2\| \sin(\alpha_{ij})} \\ &= -\frac{1}{2} \cot(\alpha_{ij}) \end{aligned}$$



and analogously $\int_{T_{ijk}} \langle \nabla\psi_i(p), \nabla\psi_j(p)\rangle dp = -\frac{1}{2}\cot(\beta_{ij})$.

Stiffness matrix - 3D, 4/4

With the same approach one can also derive the entries $\mathbf{S}_{ii}$ on the diagonal of the stiffness matrix. Eventually all the entries are given by

$$\mathbf{S}_{ij} = \begin{cases} -\frac{\cot(\alpha_{ij}) + \cot(\beta_{ij})}{2} & \text{if } (i,j) \text{ an edge} \\ -\sum_{k \neq i} \mathbf{S}_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

The stiffness matrix is sometimes also called **cotangens matrix**.

- symmetric
- positiv-semi-definit
- constant vector corresponds to 0 eigenvalue

Discrete Dirichlet energy

The idea was to find functions $\varphi_i : \mathcal{M} \to \mathbb{R}$ that are orthonormal *i.e.*$\langle \varphi_i, \varphi_j\rangle_{L^2(\mathcal{M})} = \delta_{ij}$ and minimize the **Dirichlet energy**

$$E_D(\varphi_i) = \int_{\mathcal{M}} \langle \nabla\varphi_i, \nabla\varphi_i\rangle dp = \int_{\mathcal{M}} \|\nabla\varphi_i\|^2 \, dp.$$

In the discrete case this corresponds to the optimization problem

$$\min \quad \boldsymbol{\varphi}_i^T \mathbf{S} \boldsymbol{\varphi}_i \qquad \text{s.t.} \quad \boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_j = \delta_{ij}$$

It can be shown that these functions arise as the solutions to the generalized eigenvalue problem

$$\lambda_i \mathbf{M} \boldsymbol{\varphi}_i = \mathbf{S} \boldsymbol{\varphi}_i$$

and their energies correspond to the eigenvalues λ_i.

Example

- Euclidean embeddings such as $\{\varphi_i\}_i$ can also be seen as multidimensional descriptors
- while $\mathbf{M}$ and $\mathbf{S}$ are intrinsic, the eigendecomposition has the usual problem of ambiguities
- we will identify $\mathbf{L} = \mathbf{M}^{-1}\mathbf{S}$ as the discrete **Laplace Beltrami operator**