Analysis of 3D Shapes (IN2238)

Frank R. Schmidt
Matthias Vestner

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15. Laplace Beltrami Operator

Laplace Beltrami Operator

Let $\mathcal{M}$ be a manifold and $f \in H^1(\mathcal{M})$ a function. We define $\Delta f : \mathcal{M} \to \mathbb{R}$ via

$$\int_{\mathcal{M}} \Delta f h dp = -\int_{\mathcal{M}} \langle \nabla f, \nabla h \rangle dp$$

for all *test functions* $h \in C_c^\infty(\mathcal{M})$. Δ is called the **Laplace Betrami operator (LBO)**.

- imagine $H^1(\mathcal{M})$ as the space of piecewise differentiabe functions f such that $\int_{\mathcal{M}} \langle \nabla f, \nabla f \rangle dp < \infty$
- in fact it is a so called **Sobolev space**, defined using the concept of **weak derivatives**
- $h \in C_c^\infty(\mathcal{M})$ basically means that $h \in C^\infty$ and it (and all its derivatives) vanish at the boundary of $\mathcal{M}$
- the manifolds we consider don't come with boundaries, however this definition is applicable to more general scenarios
- the LBO is a linear operator

Example: Euclidean space

Lets consider $f \in C^2(\mathbb{R})$ and $h \in C_c^\infty(\mathbb{R})$:

$$\begin{aligned}\int_{\mathbb{R}} \langle \nabla f, \nabla h \rangle dx &= \int_{\mathbb{R}} f'(x) h'(x) dx \\ &= -\int_{\mathbb{R}} \underbrace{f''(x)}_{\Delta f(x)} h(x) dx + \underbrace{\left[f'(x)h(x)\right]_{-\infty}^{\infty}}_{=0}\end{aligned}$$

Next we consider $f \in C^2(\mathbb{R}^2)$ and $h \in C_c^\infty(\mathbb{R}^2)$:

$$\begin{aligned}\int_{\mathbb{R}^2} \langle \nabla f, \nabla h \rangle dx &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\partial f}{\partial x_1} \frac{\partial h}{\partial x_1} + \frac{\partial f}{\partial x_2} \frac{\partial h}{\partial x_2} dx_1 dx_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\partial f}{\partial x_1} \frac{\partial h}{\partial x_1} dx_1 dx_2 + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\partial f}{\partial x_2} \frac{\partial h}{\partial x_2} dx_2 dx_1 \\ &= -\int_{\mathbb{R}^2} \underbrace{\left(\frac{\partial^2 f}{\partial x_1 \partial x_1} + \frac{\partial^2 f}{\partial x_2 \partial x_2}\right)}_{\Delta f(x)} h(x) dx\end{aligned}$$

LBO in local coordinates

To simplify the derivations assume that the manifold $\mathcal{M}$ can be parametrized with a single parametrization $x : U \to \mathcal{M}$, such that U does not have a boundary. In fact, a manifold without boundary must be composed of multiple charts but this technical detail will detract from the main information.

$$\begin{aligned}\int_{\mathcal{M}} \langle \nabla f, \nabla h \rangle dp &= \int_U \nabla \tilde{f} g^{-1} \nabla \tilde{h} \sqrt{\det g} du \\ &= \int_U \left(\sum_{i,j=1}^{2} \frac{\partial \tilde{f}}{\partial u_i} g^{ij} \frac{\partial \tilde{h}}{\partial u_j} \right) \sqrt{\det g(u)} du \\ &= \int_U \left(\sum_{i,j=1}^{2} \sqrt{\det g(u)} \frac{\partial \tilde{f}}{\partial u_i} g^{ij} \frac{\partial \tilde{h}}{\partial u_j} \right) du \\ &= -\int_U - \sum_{i,j=1}^{2} \frac{\partial}{\partial u_j} \left(\sqrt{\det g(u)} \frac{\partial \tilde{f}}{\partial u_i} g^{ij} \right) \tilde{h} du \\ &= -\int_U -\frac{\mathbf{1}}{\sqrt{\det \mathbf{g}(\mathbf{u})}} \sum_{\mathbf{i,j=1}}^{\mathbf{2}} \frac{\partial}{\partial \mathbf{u_j}} \left(\sqrt{\det \mathbf{g}(\mathbf{u})} \frac{\partial \tilde{\mathbf{f}}}{\partial \mathbf{u_i}} \mathbf{g^{ij}} \right) \tilde{h} \sqrt{\det g(u)} du\end{aligned}$$

LBO in local coordinates

For a function $f : \mathcal{M} \to \mathbb{R}$ defined on a $n-dimensional$ manifold one can derive an expression of Δf in local coordinates:

$$\Delta f(x(u)) = -\frac{1}{\sqrt{\det g(u)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial u_i} \left(g^{ij}(u) \frac{\partial \tilde{f}(u)}{\partial u_j} \sqrt{\det g(u)} \right)$$

where as usual $\tilde{f} = f \circ x$ and $g^{ij}(u)$ are the entries of $g^{-1}(u)$.

- it shows that the LBO is an intrinsic operator
- if g is the identity matrix, the formula boils down to the one for the euclidean case

$$\Delta f(x(u)) = \sum_{i=1}^{n} \frac{\partial^2 \tilde{f}(u)}{\partial u_i \partial u_i}$$

Discrete LBO

A discrete version of the LBO will be a linear operator from the space of piecewise linear functions to itself: $L : PL(\mathcal{M}) \to PL(\mathcal{M})$. We know that we can write such an operator as a matrix $\mathbf{L} \in \mathbb{R}^{V \times V}$ (depending on the choice of basis functions). Let $f = \sum \mathbf{f}_i \psi_i(x)$. We are looking for a function $\Delta f = \rho = \sum \boldsymbol{\rho}_i \psi_i(x)$, such that

$$\langle \rho, h \rangle = -\langle \nabla f, \nabla h \rangle \qquad \forall h = \sum \mathbf{h}_i \psi_i(x)$$

We have derived expressions for both products (mass- and stiffness- matrix):

$$\mathbf{h}^T \mathbf{M} \boldsymbol{\rho} = -\mathbf{h}^T \mathbf{S} \mathbf{f} \qquad \forall \mathbf{h} \in \mathbb{R}^V$$

Thus the coefficients $\boldsymbol{\rho}$ of Δf are given by $\boldsymbol{\rho} = -\mathbf{M}^{-1} \mathbf{S} \mathbf{f}$.
$\mathbf{L} = -\mathbf{M}^{-1} \mathbf{S}$ is called **discrete Laplace Beltrami operator**.

Helmholtz equation

The Laplacian is a formally self-adjoint operator

$$\langle \Delta f, h \rangle = -\langle \nabla f, \nabla h \rangle = \langle f, \Delta h \rangle$$

As a consequence the eigenvalue problem (Helmholtz equation)

$$\Delta \varphi_i = \lambda_i \varphi_i$$

satisfies:

- $\lambda_i \in \mathbb{R}$, in fact we can order them $0 = \lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \ldots \to -\infty$
- eigenfunctions to different eigenvalues are orthogonal $\langle \varphi_i, \varphi_j \rangle_{L^2(\mathcal{M})} = 0$.

Discrete Helmholtz equation

We can rewrite the Helmholtz equation as an equivalent **generalized eigenvalue problem**

$$\Delta\varphi_i = \lambda_i\varphi_i \Leftrightarrow \mathbf{L}\boldsymbol{\varphi}_i = \boldsymbol{\lambda}_i\boldsymbol{\varphi}_i \Leftrightarrow -\mathbf{S}\boldsymbol{\varphi}_i = \boldsymbol{\lambda}_i\mathbf{M}\boldsymbol{\varphi}_i$$

Because $\mathbf{M}$ is symmetric positive definit and $\mathbf{S}$ is symmetric, $\mathbf{L}$ is symmetric with respect to the $\mathbf{M}$-inner product:

$$\langle \mathbf{Lf}, \mathbf{g}\rangle_{\mathbf{M}} = \mathbf{f}^T\mathbf{S}\mathbf{M}^{-1}\mathbf{M}\mathbf{g} = \mathbf{f}^T\mathbf{M}\mathbf{M}^{-1}\mathbf{S}\mathbf{g} = \langle \mathbf{f}, \mathbf{Lg}\rangle_{\mathbf{M}}$$

The eigenvectors of $\mathbf{L}$ can therefore be chosen to be orthonormal with respect to the $\mathbf{M}$-inner product. If we collect all of them in a matrix $\boldsymbol{\Phi}$ this reads

$$\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \boldsymbol{I}$$

Dirichlet energy and eigenfunctions

Let $\varphi_i : \mathcal{M} \to \mathbb{R}$ be a (normalized) eigenfunction with corresponding eigenvalue λ_i and consider the Dirichlet energy

$$\int_{\mathcal{M}} \|\nabla\varphi_i\|^2 = \int_{\mathcal{M}} \langle \nabla\varphi_i, \nabla\varphi_i\rangle = -\int_{\mathcal{M}} \varphi_i\Delta\varphi_i = -\lambda_i\int_{\mathcal{M}} \varphi_i\varphi_i = -\lambda_i$$

This provides us with a nice characterization of the eigenvalues in terms of the corresponding eigenfunctions.
In particular from the above relation we see that if $\lambda_i = 0$, then φ_i must be a constant function. Further $\lambda_i = 0$ is always an eigenvalue of Δ, since $\Delta f = 0f$ for any constant function f.
All other eigenvalues are strictly negative. Sometimes the LBO (or the Helmholtz equation) is defined *with a minus*, such that all eigenvalues are positive.

Integral of eigenfunctions

For every function $f \in H^1(\mathcal{M})$ we observe

$$\int_{\mathcal{M}} \Delta f = \int_{\mathcal{M}} 1\Delta f = -\int_{\mathcal{M}} \langle \nabla 1, \nabla f\rangle = 0$$

This implies that for every eigenfunction φ_i with corresponding eigenvalue $\lambda_i \neq 0$:

$$\int_{\mathcal{M}} \varphi_i = \frac{1}{\lambda_i}\int_{\mathcal{M}} \Delta\varphi_i = 0$$

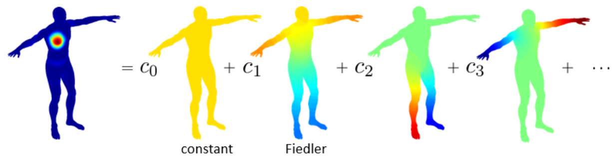
The eigenfunction φ_0 with corresponding eigenvalue $\lambda_0 = 0$ is constant, $\varphi_0(p) = c$. Due to the normalization we get

$$1 = \int_{\mathcal{M}} \varphi^2(p)dp = c^2\text{area}(\mathcal{M}) \qquad \Leftrightarrow c = \pm\frac{1}{\sqrt{\text{area}(\mathcal{M})}}$$

Change of basis

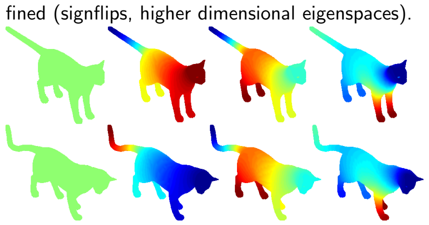
Due to the orthogonality of eigenfunctions, we can write every function $f \in L^2(\mathcal{M})$ as a linear combination

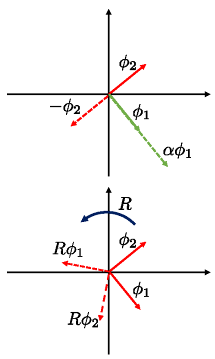
$$f(x) = \sum_{i=1}^{\infty} c_i\varphi_i(x) = \sum_{i=1}^{\infty}\langle f, \varphi_i\rangle_{L^2}\varphi_i(x)\mathbf{f} \qquad = \sum_i \langle \mathbf{f}, \boldsymbol{\varphi}_i\rangle_{\mathbf{M}}\boldsymbol{\varphi}_i = \boldsymbol{\Phi}\boldsymbol{\Phi}^T\mathbf{M}\mathbf{f}$$



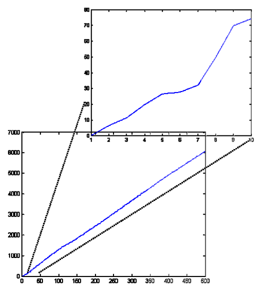
Invariance under isometries

The Laplace Beltrami operator is an intrinsic operator and therefore invariant to isometric deformations.
However the eigenfunctions are not uniquely defined (signflips, higher dimensional eigenspaces).





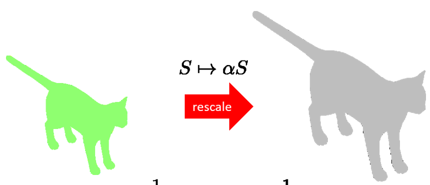
Weyl's law



$$|\lambda_j| \sim \frac{\pi}{\text{area}(\mathcal{M})}j$$

Influence of scaling

What happens to the eigenvalues and eigenfunctions when we simply rescale a shape?
Weyls law is already suggesting us that something is going to change.



$$\phi \mapsto \frac{1}{\alpha}\phi \qquad \lambda \mapsto \frac{1}{\alpha^2}\lambda$$

Spectral descriptors

We will consider different descriptors that rely on the LBO.