# Analysis of 3D Shapes (IN2238)

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### Summary Laplacian

The Laplace Beltrami operator is an operator matching a function $f : \mathcal{M} \to \mathbb{R}$ to another function $\Delta f : \mathcal{M} \to \mathbb{R}$.

$$\int_{\mathcal{M}} \Delta f h dp = -\int_{\mathcal{M}} \langle \nabla f, \nabla h \rangle dp \qquad \forall h$$

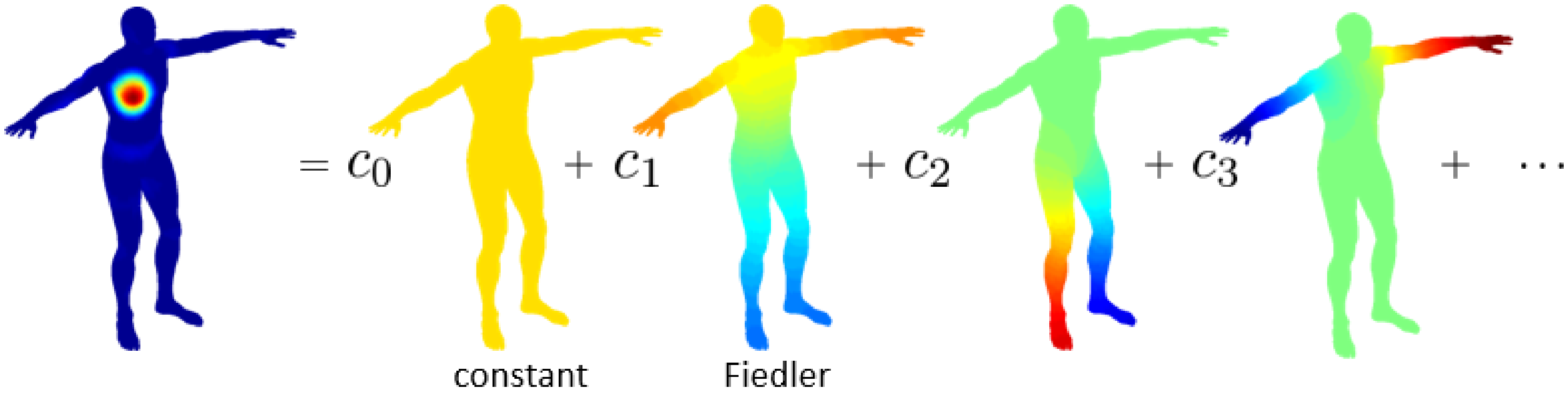
Some key properties are:

- it is a linear operator
- it is an **intrinsic** operator
- it is **(formally) self-adjoint**
- it has a discrete spectrum (countably many eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$)
- all eigenvalues are real and non-positive
- the first eigenvalue (when ordered with increasing magnitude) equals $0$. The corresponding eigenfunction is constant
- the eigenfunctions can be chosen to be orthonormal

## Change of basis

Due to the orthogonality of eigenfunctions, we can write every function $f \in L^2(\mathcal{M})$ as a linear combination

$$f(x) = \sum_{i=1}^{\infty} c_i \varphi_i(x) = \sum_{i=1}^{\infty} \langle f, \varphi_i \rangle_{L^2} \varphi_i(x) \mathbf{f} \qquad = \sum_i \langle \mathbf{f}, \boldsymbol{\varphi}_i \rangle_{\mathbf{M}} \boldsymbol{\varphi}_i = \mathbf{\Phi}\mathbf{\Phi}^T \mathbf{M}\mathbf{f}$$



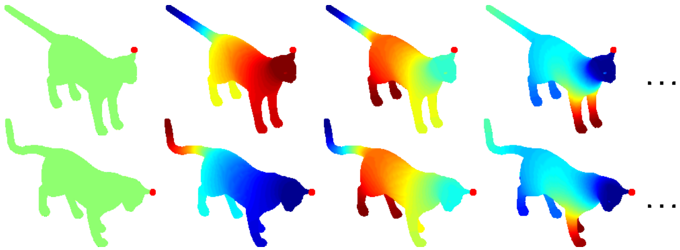
## Spectral descriptors

We will consider different descriptors that rely on the LBO.

## Global point signature

The most straightforward approach is to map each point $p \in \mathcal{M}$ to an infinite-dimensional vector according to the eigenfunctions of the Laplacian:

$$p \mapsto (\varphi_1(p), \varphi_2(p), \varphi_3(p), \dots) \in \mathbb{R}^\infty$$



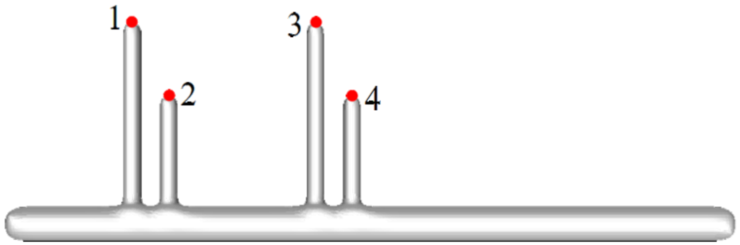
A scale invariant version is the **global point signature (GPS)**

$$p \mapsto \left(\frac{\varphi_2(p)}{\sqrt{|\lambda_2|}}, \frac{\varphi_3(p)}{\sqrt{|\lambda_3|}}, \frac{\varphi_4(p)}{\sqrt{|\lambda_4|}}, \dots\right) \in \mathbb{R}^\infty$$

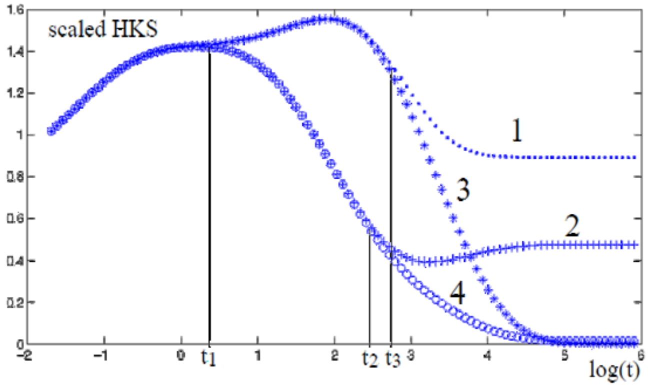
It weights **lower frequencies** with a bigger weight.

**Multiscale property**

In general it would be desirable to have a descriptor which captures geometric information at **different scales**.




For small scales (locally) points $1$ and $3$ are not distinguishable.
The notion of scale we are looking for should provide a desciptor having an analogous behaviour to the one depicted in the following figure



## Heat diffusion

The diffusion of heat on a surface can be described by the **heat equation**:

$$\frac{\partial u(x,t;u_0)}{\partial t} = \Delta u(x,t;u_0)$$

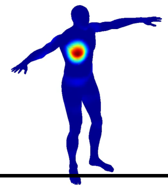
We write $u(x,t;u_0)$ for the amount of heat at point $x$ after time $t$, when at time zero the distribution of heat is given by

$$u(x,0) = u_0(x)$$

**Physical interpretation:**

Rate of change of heat within a region $V$ equals the negative of the flux through $\partial V$:

$$\frac{d}{dt}\int_V u dx = -\int_{\partial V} \mathbf{F}\cdot \nu dS = -\int_V \mathrm{div}(\mathbf{F})$$



In many situations $\mathbf{F}$ is proportional to the gradient of $u$ but points in the opposite direction: $\mathbf{F} = -a\nabla u$

## Dirac disctribution

The **Dirac distribution** placed at a point $z \in \mathcal{M}$ is defined via

$$\langle f, \delta_z \rangle = \int_{\mathcal{M}} f(x)\delta_z(x)dx = f(z)$$

for all continuous functions $f$.

- the dirac distribution is not a function in the classical sense
- what is written above is not really an inner product but a so called **dual pairing**
- on $\mathbb{R}$ you can imagine a dirac as the limit of thinner and thinner gaussians

$$\delta_z(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{\|x-z\|^2}{2\sigma^2})$$

- the discrete approximation of the dirac is given by $\boldsymbol{\delta}_z = \mathbf{M}^{-1}\mathbf{e}_z$



## Heat kernel

A soloution to the heat equation is given by

$$u(x, t; u_0) = \int_{\mathcal{M}} k_t^{\mathcal{M}}(x, y)u_0(y)dy$$

The function $k_t^{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is called **heat kernel** of $\mathcal{M}$ and it is describes from one point to another in time $t$.
In particular assume we want to diffuse heat from a dirac distribution $\delta_z$. In that case we get:

$$u(x, t; \delta_z) = \int_{\mathcal{M}} k_t^{\mathcal{M}}(x, y)\delta_z(y)dy = k_t^{\mathcal{M}}(x, z)$$

## Heat kernel euclidean

The heatkernel in $\mathbb{R}^n$ is given by

$$k_t^{\mathbb{R}^n}(x,y) = \frac{1}{\sqrt{4\pi t}^n} \exp(-\frac{\|x-y\|^2}{4t})$$

From the expression above we see that the distance between two points (Euclidean in this case) can be recovered from the heat kernel.

The Dirac distribution can be modelled as the limit

$$\delta_z(\cdot) = \lim_{t \to 0} k_t(z, \cdot)$$



## Heat kernel on manifolds

Also the geodesic distance ot two points on a two dimensional manifold can be recovered from the heat kernel
**Varadhan's formula:**

$$d^2(x,y) = -\lim_{t \to 0} 4t \log(k_t^{\mathcal{M}}(x,y))$$

In addition we have the following **informative property**:
$\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ is an isometry iff

$$k_t^{\mathcal{M}_1}(x,y) = k_t^{\mathcal{M}_2}(\Phi(x), \Phi(y))$$

## Solving the heat equation using eigenfunctions of the LBO 1

$$\frac{\partial u(x,t;u_0)}{\partial t} = \Delta u(x,t;u_0) \qquad\qquad u(x,0) = u_0(x)$$

We know that the eigenfunctions of the LBO $\Delta$ form a basis, thus for every $t$ we can find coefficients $c_k(t)$, such that

$$u(t,x;u_0) = \sum_{k=1}^{\infty} c_k(t)\varphi_k(x)$$

Using the linearity of $\frac{\partial}{\partial t}$ and $\Delta$ the heat equation becomes

$$\sum_{k=1}^{\infty} \dot{c}_k(t)\varphi_k(x) = \sum_{k=1}^{\infty} c_k(t)\Delta\varphi_k(x) = \sum_{k=1}^{\infty} \lambda_k c_k(t)\varphi_k(x)$$

## Solving the heat equation using eigenfunctions of the LBO 2

We have transformed the heat equation

$$\frac{\partial u(x,t;u_0)}{\partial t} = \Delta u(x,t;u_0) \qquad\qquad u(x,0) = u_0(x)$$

into a collection of ordinary differential equations:

$$\dot{c}_k(t) = \lambda_k c_k(t) \forall k$$

with solutions $c_k(t) = d_k \exp(\lambda_k t)$.
The coefficients $d_k$ are determined by the initial distribution of heat:

$$u(0,x;u_0) = \sum_{k=1}^{\infty} d_k \varphi_k(x) = u_0(x) \qquad\qquad \Rightarrow d_k = \langle \varphi_k, u_0 \rangle$$

$$u(t,x;u_0) = \sum_{k=1}^{\infty} \langle \varphi_k, u_0 \rangle \exp(\lambda_k t) \varphi_k(x)$$

## Heat kernel using eigenfunctions

The heat kernel $k_t^{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ was defined as the solution of the heat equation when initialized with a dirac:

$$k_t^{\mathcal{M}}(x,y) = u(x,t;\delta_y)$$

In terms of eigenfunctions this gives:

$$\begin{aligned} k_t^{\mathcal{M}}(x,y) &= \sum_{k=1}^{\infty} \langle \varphi_k, \delta_y \rangle \exp(\lambda_k t) \varphi_k(x) \\ &= \sum_{k=1}^{\infty} \exp(\lambda_k t) \varphi_k(y) \varphi_k(x) \end{aligned}$$



## Heat kernel signature

We define the **heat kernel signature** (HKS) at a point $x \in \mathcal{M}$ as the vector

$$hks(x) = (k_{t_1}(x,x), \ldots, k_{t_T}(x,x)) \in \mathbb{R}^T$$

$$k_t(x,x) = \sum_{k=0}^{\infty} e^{\lambda_k t} \varphi_k^2(x)$$

In this view, each evaluation of the heat kernel in the vector above describes **the amount of heat staying at point** $x$ after time $t$, when starting with a unit heat source (dirac) at $x$ itself.

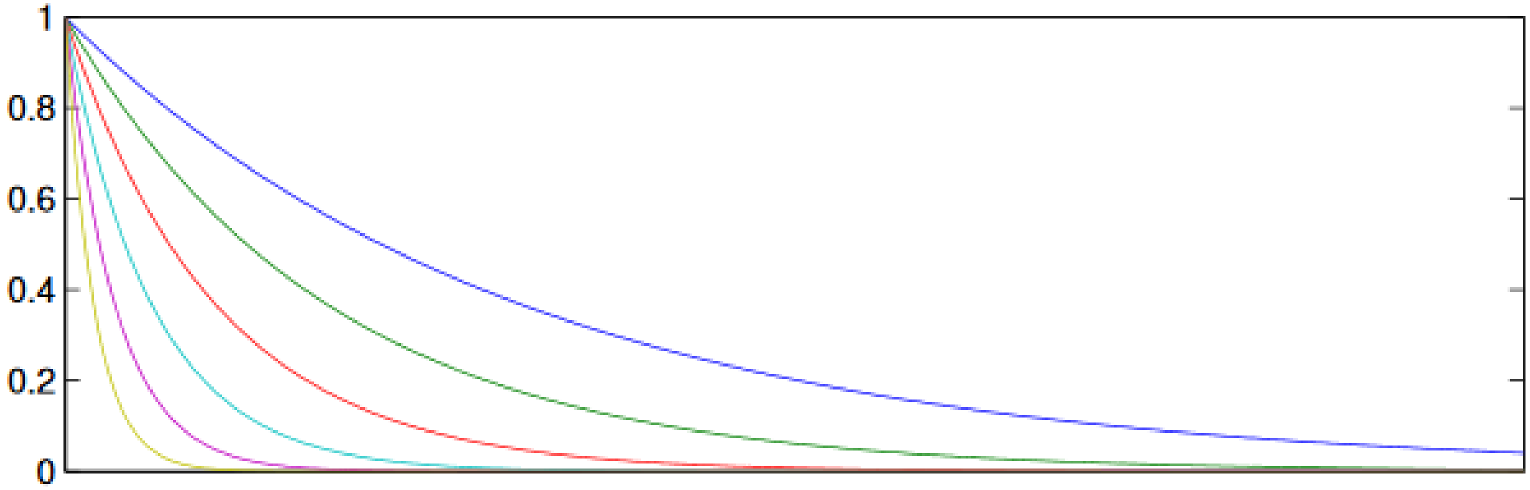
The HKS also has an informative property. If the eigenvalues of the Laplacians on $\mathcal{M}_1$ and $\mathcal{M}_2$ are not repeated, then:
$\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ is an isometry iff $k_t^{\mathcal{M}_1}(x,x) = k_t^{\mathcal{M}_2}(\Phi(x), \Phi(x))$

## HKS as low pass filter

The entries of the HKS can be seen as a weighted sum of $\varphi_k^2(x)$:

$$\sum_{k=0}^{\infty} f(\lambda_k; t)\varphi_k^2(x)$$

where $f(\lambda; t) = \exp(\lambda t)$ is a **low-pass filter** for all parameters $t$.

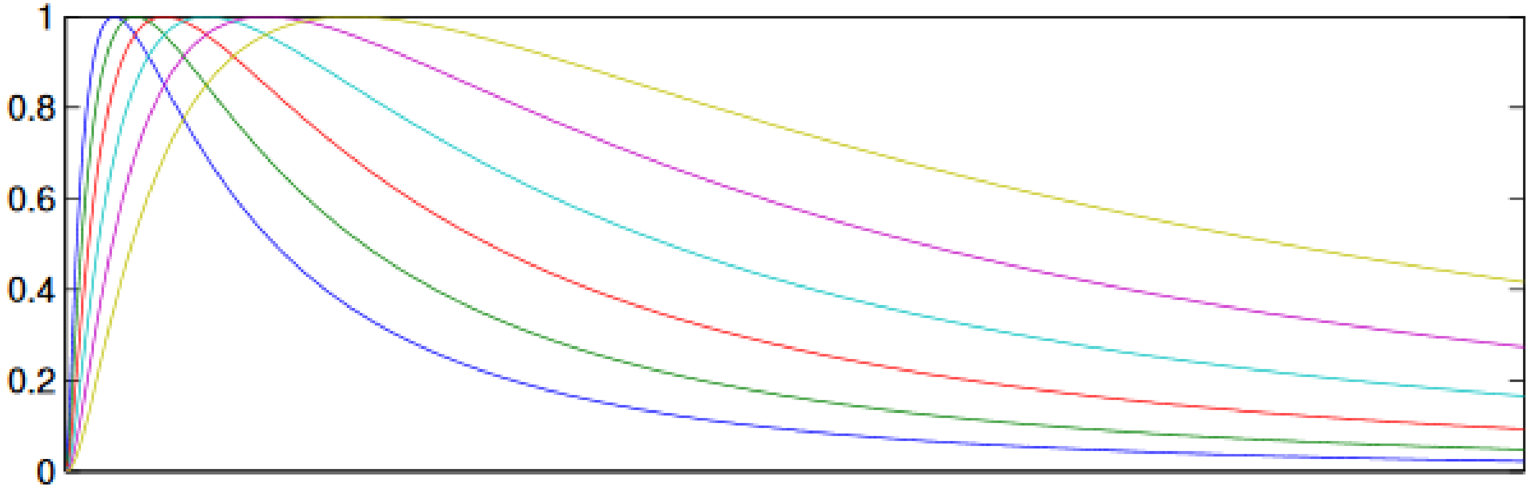


## WKS as band pass filter

In contrast the **wave kernel signature** (WKS) chooses $f$'s that act as band pass filters

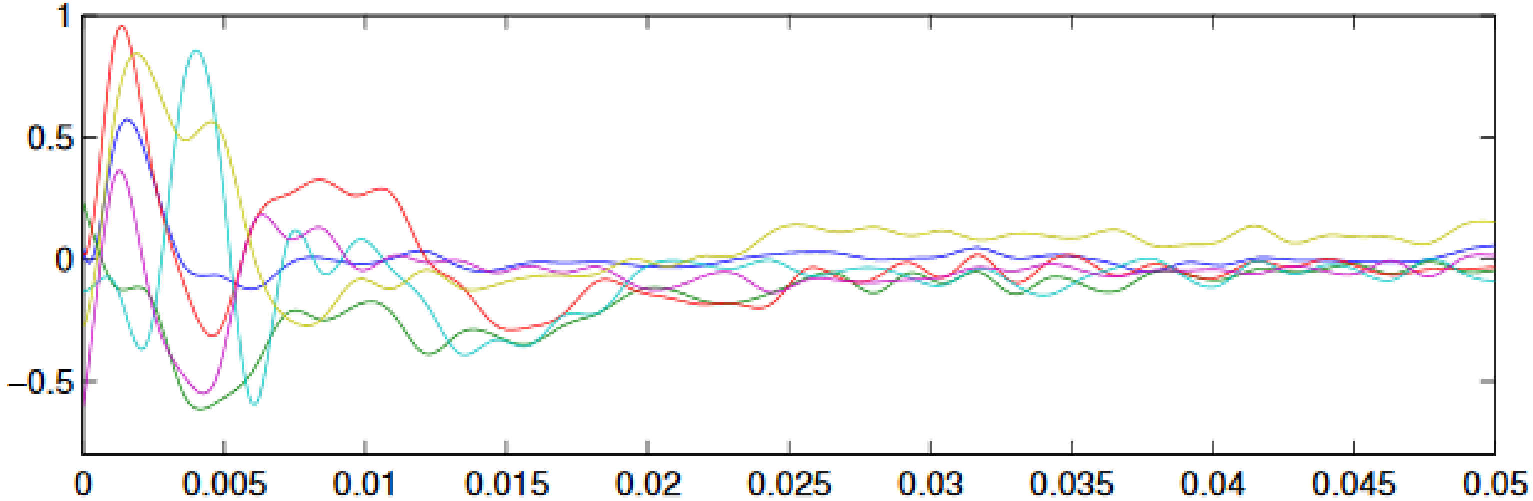
$$\sum_{k=0}^{\infty} f(\lambda_k; t)\varphi_k^2(x)$$

$$f(\lambda, t) = \exp(-\frac{(\log(\lambda_k) - t)^2}{2\sigma^2})$$



## Learned filter

At some point people started to learn the weighting funcions $f$:






## Literature


- Sun, Jian, Maks Ovsjanikov, and Leonidas Guibas. *A concise and provably informative multiscale signature based on heat diffusion.* Computer graphics forum. Vol. 28. No. 5. Blackwell Publishing Ltd, 2009.
- Aubry, Mathieu, Ulrich Schlickewei, and Daniel Cremers. *The wave kernel signature: A quantum mechanical approach to shape analysis.* Computer Vision Workshops (ICCV Workshops), 2011 IEEE International Conference on. IEEE, 2011.
- Litman, Roee, and Alexander M. Bronstein. *Learning spectral descriptors for deformable shape correspondence.* IEEE transactions on pattern analysis and machine intelligence 36.1 (2014): 171-180.
- Windheuser, Thomas, et al. *Optimal Intrinsic Descriptors for Non-Rigid Shape Analysis.* BMVC. 2014.