# ANALYSIS OF THREE-DIMENSIONAL SHAPES FUNCTIONAL MAPS 

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## SHAPE MATCHING

וחנח


We already saw for 2D Matchings that a correspondence can be represented as permutation matrices if both shapes have the same number of vertices.

$\min _{P} E(P)$
s.t. $P$ is permutation

This does not scale well with the size of the shapes

## PROBLEMS




$$
x
$$

## FUNCTIONAL MAP

Assume we were given a bijection $T: M \rightarrow N$. Given any scalar function $f: M \rightarrow \mathbb{R}$ on $M$ we can induce $g: N \rightarrow \mathbb{R}$ by composition.


We can denote this transformation by a functional $T_{F}$ such that

$$
T_{F}(f)=f \circ T^{-1}
$$

## NO INFORMATION LOSS

If we know $T$, we can obviously construct $T_{F}$ by its definition $T_{F}(f)=f \circ T^{-1}$

Can we also reconstruct $T$ if we only know $T_{F}$ ? Yes, we can!
Let $\delta_{a}: M \rightarrow \mathbb{R}$ be an indicator function on $M$ such that

$$
e_{a}(x)= \begin{cases}1 & \text { if } x=a \\ 0 & \text { otherwise }\end{cases}
$$

Then if we call $g=T_{F}\left(e_{a}\right)$, it must be $g(y)=\left(e_{a} \circ T^{-1}\right)(y)=0$ whenever $T^{-1}(y) \neq a$ and $g(y)=1$ otherwise.
Since $T$ is a bijection, this happens only once and $T(a)$ is the unique point $y \in N$ such that $g(y)=1$

We can show that $T_{F}$ is a linear map:

$$
\begin{aligned}
T_{F}(\alpha f+g) & =(\alpha f+g) \circ T^{-1} \\
& =\alpha f \circ T^{-1}+g \circ T_{(\text {by linearity of composition) }}^{-1} \\
& =\alpha T_{F}(f)+T_{F}(g)
\end{aligned}
$$

The key observation is that, while $T$ can be a very complex transformation, $T_{F}$ always acts linearly.

This means we can give $T_{F}$ a matrix representation after choosing a basis for two function spaces on $M$ and $N$.

Let $\left\{\phi_{i}^{M}, \phi_{j}^{N}\right\}$ be bases for function spaces $\mathcal{F}(M), \mathcal{F}(N)$ on $M, N$ such that $f=\sum_{i} a_{i} \phi_{i}^{M}, f \in \mathcal{F}(M)$. Then we can write:

$$
T_{F}(f)=T_{F}\left(\sum_{i} a_{i} \phi_{i}^{M}\right)=\sum_{i} a_{i} T_{F}\left(\phi_{i}^{M}\right)
$$

$$
T_{F}\left(\phi_{i}^{M}\right)=\sum_{j} c_{j i} \phi_{j}^{N}
$$

Putting both together, we get:

$$
T_{F}(f)=\sum_{i} a_{i} \sum_{j} c_{j i} \phi_{j}^{N}=\sum_{j, i} a_{i} c_{j i} \phi_{j}^{N}
$$

$$
\begin{aligned}
T_{F}(f) & =\sum_{j} \sum_{i} c_{j} \phi_{j}^{N} \\
& =\sum_{j} b_{j} \phi_{j}^{N}
\end{aligned}
$$

We can represent each function $f$ on $M$ by its coefficients $a_{i}$, and similarly $T_{F}(f)$ on $N$ by the coefficients $b_{j}$.

Rewriting in matrix notation, we have:

$$
T_{F}(a)=b=C a
$$

If the bases are orthogonal with respect to some inner product $\langle\cdot, \cdot\rangle$, then we can simply write

$$
a_{i}=\left\langle f, \phi_{i}^{M}\right\rangle \quad c_{i j}=\left\langle T_{F}\left(\phi_{i}^{M}\right), \phi_{j}^{N}\right\rangle
$$

Lets take a closer look at $c_{i j}=\left\langle T_{F}\left(\phi_{i}^{M}\right), \phi_{j}^{N}\right\rangle$
We know it holds: $\quad P e_{x}=e_{T(x)} \quad$ Indicator function for vertex $T(x) \in N$
Indicator function for vertex $x \in M$
$a=\Phi_{M}^{-1} e_{x} \quad$ Indicator function in the chosen basis of M
$C a \quad$ Indicator function mapped to the basis of N
$\Phi_{N} C a \quad$ Indicator function on N in the indicator basis

$$
\begin{aligned}
& \Phi_{N} C \Phi_{M}^{11} e_{x}=e_{T(x)} \\
& \Phi_{N} C \Phi_{M}^{-1}=P
\end{aligned} \quad C=\Phi_{N}^{-1} P \Phi_{M}
$$

Lets take a closer look at

$$
C=\Phi_{N}^{-1} P \Phi_{M}
$$

each column is an eigenfunction

Simply put, the Functional map C contains all the inner products between the basis functions of the two shapes, after the vertex ordering has been disambiguated by the bijection P.

This relation was actually already visible here:

$$
c_{i j}=\left\langle T_{F}\left(\phi_{i}^{M}\right), \phi_{j}^{N}\right\rangle
$$



Up until now we have been assuming the presence of a basis for functions defined on the two shapes. The first possibility is to consider the indicator basis on each shape:

$$
\begin{aligned}
& \phi_{i}^{M}(x)= \begin{cases}1 & , x=i \\
0 & , \text { otherwise }\end{cases} \\
& C=\Phi_{N}^{-1} P \Phi_{M} \\
& C=P
\end{aligned}
$$

$$
C a=b \quad \square \quad \begin{aligned}
& P a=b \\
& P \text { permutation matrix }
\end{aligned}
$$



## Wero eigenfunction basis

But we already learned about another possibility last week!
The eigenfunctions of the Laplace-Beltrami operator form an orthogonal basis (w.r.t. the weighted inner product $\langle\cdot, \cdot\rangle_{M}$ ) for $l^{2}$ functions on each shape.

In particular, we can approximate:

$$
f=\sum_{i=0}^{\infty} a_{i} \phi_{i}^{M} \approx \sum_{i=0}^{m} a_{i} \phi_{i}^{M}
$$


$m=200$

$m=100$

$m=50$

## WiflBO EIGENFUNCTION BASIS

This means we can also approximate:

$$
f=\sum_{i, j=0}^{\infty} a_{i} c_{i j} \phi_{j}^{N} \approx \sum_{i, j=0}^{m} a_{i} c_{i j} \phi_{j}^{N}
$$

Looking at matrix notation, we are reducing the size of $C$ to $m \times m$


LBO eigenfunctions


$$
m \ll n
$$

## STRUGTURE IN C

Isometries:


Non-Isometries:


## EXAMPLES


(a) source

(b) ground-truth map

(c) left to right map

(d) head to tail map

Note that not every linear map corresponds to a (bijective) point-to-point correspondence.

If we know enough compatible functions $a$ and $b$ we can deduce the linear relation by solving a bunch of linear equations:

$$
C a=b
$$

## Descriptor preservation

If we are given $k$ descriptors, we can phrase $k$ equations:

$$
\begin{array}{ll}
C a_{1}=b_{1} & \text { For instance, consider } \\
C a_{2}=b_{2} & \text { curvature or the Heat }
\end{array}
$$

$$
C a_{k}=b_{k}
$$

## Landmark matches

Assume we know $T(x)=y$ for some $x$. We can calculate the geodesic distance maps on both shapes and use them as constraints:

$$
d_{x}^{M}\left(x^{\prime}\right)=d_{M}\left(x, x^{\prime}\right)=a \quad d_{y}^{N}\left(y^{\prime}\right)=d_{N}\left(y, y^{\prime}\right)=b
$$

$$
C\left(\begin{array}{ccc}
\mid & & \mid \\
\mathbf{a}_{1} & \cdots & \mathbf{a}_{n} \\
\mid & & \mid
\end{array}\right)=\left(\begin{array}{ccc}
\mid & & \mid \\
\mathbf{b}_{1} & \cdots & \mathbf{b}_{n} \\
\mid & & \mid
\end{array}\right) \quad \begin{array}{ll}
n<m & \text { under-determined } \\
& n=m \\
\text { full rank } \\
n>m & \text { over-determined }
\end{array}
$$

$m \times m$
$m \times n$
$m \times n$

In the common case in which $n>m$, we can solve the resulting linear system in the least-squares sense:

$$
C A=B \Rightarrow C^{*}=\arg \min _{C}\|C A-B\|_{2}^{2}
$$

## IMPOSING STRUCTURE

If we have prior knowledge about the structure of the map we can also add regularizers to the optimization term:

$$
\arg \min _{C}\|C A-B\|_{2}^{2}+\rho(C)
$$

For example, diagonal structure for isometries:


$$
\arg \min _{C}\|C A-B\|_{2}^{2}+\|C \circ W\|_{2}
$$




W

Imagine the blue line is diagonal...

Once we have found an optimal Functional Map $C^{*}$, we may want to convert it back to a point-to-point correspondence.

Simplest idea: Map indicator functions at each point.


This is very inefficient and sensitive to numerical errors from truncation.

## WifUNCTIONS TO CORRESPONDENCE

Observe that each indicator function around $\underline{x}$. when represented in the eigenbasis, has as coefficients the $k$-th column of the matrix $\Phi_{M}$ where $k$ is the index of point x.

$$
\Phi_{M}^{\top} e_{k} \in \mathbb{R}^{m}
$$

Representation of one indicator function in the eigenbasis

$$
\Phi_{M}^{\top} \in \mathbb{R}^{m \times n}
$$

Representation of all indicator functions in the eigenbasis
$\Phi_{M}^{\top}$ can be regarded as a set of $n$ points in $\mathbb{R}^{m}$


## WFUNCTIONS TO CORRESPONDENCE

Clearly, the same can be done for the eigenfunctions on the second shape $N$

$$
\begin{aligned}
& \begin{array}{l}
\bullet \mathbb{R}^{m} \quad \bullet \quad \bullet \quad \begin{array}{l}
\text { We can find correspondences by } \\
\text { aligning both point clouds and } \\
\text { searching for nearest neighbors }
\end{array} \\
\Phi^{\bullet} y^{*}=\arg \min _{y}^{\top}\left\|T_{F}\left(e_{x}\right)-e_{y}\right\|_{L^{2}}^{2}
\end{array} \\
& \approx \arg \min _{y}\left\|C\left(\begin{array}{c}
\phi_{1}^{M}(x) \\
\vdots \\
\phi_{m}^{M}(x)
\end{array}\right)-\left(\begin{array}{c}
\phi_{1}^{N}(y) \\
\vdots \\
\phi_{m}^{N}(y)
\end{array}\right)\right\|_{L^{2}}^{2} \\
& \min _{P \in\{0,1\}^{n \times n}}\left\|C\left(\Phi^{M}\right)^{\top}-\Phi^{N} P\right\|_{F}^{2} \\
& \text { s.t. } \quad P^{\top} 1=1 \\
& P 1=1
\end{aligned}
$$

## EXAMPLE



Recovering correspondences from low-rank Functional Maps is a whole problem on its own.

Even when choosing a small k, a lot of compatible functions are necessary to reliably solve for C without imposing any regularization.

But the regularization terms heavily depend the basis (and assumptions about the shapes).

Laplace-Beltrami eigenbasis: robust to nearly-isometric deformations only!

The recovered correspondences are often neither bijective nor continuous.

Matching Partial Shapes


Moving away from isometries


Combining with Neural Networks


Extensions for vector fields...

- Functional Maps: A Flexible Representation of Maps Between Shapes.

Ovsjanikov, Ben-Chen, Solomon, Butscher, Guibas. ACM SIGGRAPH, 2012.

- Sparse Modeling of Intrinsic Correspondences. Pokrass, Bronstein, Bronstein, Sprechmann, Sapiro. Computer Graphics Forum, 2014.
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Correspondence. Litany, Remez, Rodolà, Bronstein, Bronstein. arXiv:1704.08686, 2017.

