Analysis of Three-Dimensional Shapes
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Summer Semester 2017

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Weekly Exercises 1

Room: 02.09.023

Wed, 17.05.2017, 14:00-16:00

Submission deadline: Tue, 16.05.2017, 23:59 to laehner@in.tum.de

Mathematics: Calculus recap and Manifolds

Recap the definition of *partial derivative* if you are not familiar with it anymore. Quick introduction of notation: For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ the partial derivative of the $j - th$ component of f by the i-th variable can be written as

1. $\partial_i f^j$ with $i \in \{1, ..., n\}, j \in \{1, ..., m\}$
2. $\frac{\partial f^j}{\partial x_i}$ describing the same thing but assuming that the variable are given names as is normally case (e.g. $(x, y, z) \mapsto (x, y + z)$)

The notation is a matter of taste but some are less confusion depending on the situation.

The *differential* is the best linear approximation of a function. For a function $f : \mathbb{R}^n \to \mathbb{R}^m$ it can be represented by its Jacobi matrix:

$$Df = \begin{pmatrix} \partial_1 f^1 & \dots & \partial_n f^1 \\ \vdots & & \vdots \\ \partial_1 f^m & \dots & \partial_n f^m \end{pmatrix}$$

or (if taking partial derivatives is not trivial)

$$Df(x)[h] \doteq f(x + h) - f(x)$$

In this case the equality holds only for linear terms in h.

Exercise 1 (2 points). 1. Let f be

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x^2+y^2} & \text{otherwise} \end{cases} \end{aligned}$$

Calculate the partial derivatives $\partial_1 f$ and $\partial_2 f$. What happens at $\partial_1 f(0, 0)$?

2. Consider $g:\ \mathbb{R}^2 \to \mathbb{R}$ and $f:\ \mathbb{R}^2 \to \mathbb{R}$ differentiable with

$$g(x_1, x_2) = f(x_1^2, x_1 + x_2)$$

Calculate $\frac{\partial g}{\partial x_1}$ (in relation to f).

Solution. 1. For $(x, y) \neq 0$:

$$\begin{aligned} \partial_x f(x,y) &= \frac{y^3 - x^2 y}{x^4 + 2x^2y^2 + y^4} \\ \partial_y f(x,y) &= \frac{x^3 - y^2 x}{x^4 + 2x^2y^2 + y^4} \end{aligned}$$

$\partial_x f(0,0) = 0$ because $f(x,0) = 0$ and therefore $\lim_{x\to 0} f(x,0) = 0$.

2. Solution for each notation.

(a) In this case there are two different scopes for x_1.

$$\begin{aligned} \frac{\partial g}{\partial x_1} &= \frac{\partial f(x_1^2, x_1 + x_2)}{\partial x_1}\frac{dx_1^2}{dx_1} + \frac{\partial f(x_1^2, x_1 + x_2)}{\partial x_2}\frac{dx_1 + x_2}{dx_1} \\ &= \frac{\partial f(x_1^2, x_1 + x_2)}{\partial x_1} \cdot 2x_1 + \frac{\partial f(x_1^2, x_1 + x_2)}{\partial x_2} \cdot 1 \end{aligned}$$

(b)

$$\begin{aligned} \partial_1 g &= \partial_1 f(x_1^2, x_1 + x_2)\frac{dx_1^2}{dx_1} + \partial_2 f(x_1^2, x_1 + x_2)\frac{dx_1 + x_2}{dx_1} \\ &= \partial_1 f(x_1^2, x_1 + x_2) \cdot 2x_1 + \partial_2 f(x_1^2, x_1 + x_2) \cdot 1 \end{aligned}$$

(c) You can also read out the partial derivative from the differential. We define $h : \mathbb{R}^2 \to \mathbb{R}^2, h(x_1, x_2) = (x_1^2, x_1 + x_2)$ and use the chain rule.

$$\begin{aligned} (Dg)_{(x)} \cdot e_1 &= (Df \circ h)_{(x)} \cdot e_1 \\ &= (Df)_{(h(x))} \cdot (Dh)_{(x)} \cdot e_1 \\ &= \big(((\partial_1 f) \circ h)(x) \quad ((\partial_2 f) \circ h)(x)\big) \begin{pmatrix} \partial_1 h^1(x) & \partial_2 h^1(x) \\ \partial_1 h^2(x) & \partial_2 h^2(x) \end{pmatrix} e_1 \\ &= \big(((\partial_1 f) \circ h)(x) \quad ((\partial_2 f) \circ h)(x)\big) \begin{pmatrix} 2x_1 & 0 \\ 1 & 1 \end{pmatrix} e_1 \\ &= ((\partial_1 f)(h(x)) \cdot 2x_1 + ((\partial_2 f)(h(x)) \end{aligned}$$

Exercise 2 (2 points). 1. Calculate the differential of

$$\begin{aligned} f_1 : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (x(1 - y), xyz) \end{aligned}$$

2. Calculate the differential of

$$\begin{aligned} f_2 : \mathbb{R}^2 &\to \mathbb{R}^3 \\ (u, v) &\mapsto (u^2 + v^2, u - v, 4v^4) \end{aligned}$$

Solution. 1.

$$Df_1 = \begin{pmatrix} 1-y & -x & 0 \\ yz & xz & xy \end{pmatrix}$$

2.

$$Df_2 = \begin{pmatrix} 2u & 2v \\ 1 & -1 \\ 0 & 16v^3 \end{pmatrix}$$

Exercise 3 (3 points)**.** Consider the vector spaces $U = \mathbb{R}^3$ and $V = \mathbb{R}^2$ which can be equipped with the canonical basis C_3, C_2 or the following ones:

$$X_1 = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \qquad X_2 = \left(\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 10 \\ 10 \end{pmatrix} \right)$$

$$Y_1 = \left(\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} \right) \qquad Y_2 = \left(\begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right)$$

Let $L : U \to V$ be a linear mapping that can be represented by $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ in the canonical basis.

1. Write down $\mathcal{M}_{Y_1}^{X_1}(L)$.
2. Write down $\mathcal{M}_{Y_2}^{X_2}(L)$.
3. You have $a \in U$ written in the basis X_1 with the coefficients $(1, 1, 1)$. What is the result of applying L to a written in the canonical basis?

Tip: no need to calculate the matrix inverses by hand.

Solution. Note: in the first version of the slides the formula for $\mathcal{M}_Y^X(L)$ was wrong.

Notice that A is written in the canonical basis, so actually $A = (M)_{C_2}^{C_3}(L)$ and a basis transformation from/to B is a form of a linear mapping of the identity. You could write it as $\mathcal{M}_B^{C_n}(Id) = B^{-1}$ and $\mathcal{M}_{C_n}^B(Id) = B$. You cannot simply apply the formula from lecture because there A was given in some non-canonical basis and here the input and outputs are supposed to be non-canonical.

1.

$$\mathcal{M}_{Y_1}^{X_1}(L) = Y_1^{-1} \cdot A \cdot X_1 \qquad = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 & 6 \\ 2 & 2 & 0 \end{pmatrix}$$

2.

$$\mathcal{M}_{Y_2}^{X_2}(L) = Y_2^{-1} \cdot A \cdot X_2 \qquad = \begin{pmatrix} 0.\bar{1} & 0.\bar{2} \\ 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 & 10 \\ 0 & 2 & 10 \\ 0 & 0 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 1.\bar{3} & 0.\bar{6} & 10 \\ 2 & 0 & 0 \end{pmatrix}$$

3. We need to calculate $(M)_{C_2}^{X_1}(L) = A \cdot X_1 = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 1 & 3 \end{pmatrix} = B$. And $B \cdot a^\top = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$.