Analysis of Three-Dimensional Shapes
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Weekly Exercises 5

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Mathematics: First Fundamental Form

Exercise 1 (2 points). Consider the following coordinate maps from Exercise sheet 3:

$$c_i : [0,1] \to \mathbb{R}^2$$

$$c_1 : t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \qquad c_2 : t \mapsto \begin{pmatrix} \cos(t^2) \\ \sin(t^2) \end{pmatrix}$$

Calculate the first fundamental form of both parametrizations.

Solution.

1. $Dc_1 = \begin{pmatrix} -sin(t) \\ cos(t) \end{pmatrix}$

$$Dc_1^\top Dc_1 = sin(t)^2 + cos(t)^2 = 1 = ||c_1'(t)||^2$$

2. $Dc_2 = \begin{pmatrix} -sin(t^2) \cdot 2t \\ cos(t^2) \cdot 2t \end{pmatrix}$

$$Dc_2^\top Dc_2 = 4sin(t^2)^2 t^2 + 4cos(t^2)^2 t^2 = ||c_2'(t)||^2$$

Exercise 2 (3 points). Consider the coordinate map (from Exercise Sheet 2, $C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$):

$$x : C \to \mathbb{R}^3$$
$$(u,v) \mapsto (u, v, \sqrt{1 - u^2 - v^2})$$

1. Calculate the first fundamental form for each point on the surface.
2. Integrate the length of the straight line l_1 between $(0,0) \in C$ and $(0, 0.8) \in C$ both on C and the manifold.
3. Let l_2 be the line between $(0,0)$ and $(0.565685, 0.565685)$. Calculate the angle between l_1, l_2 on the domain and on the manifold.

Solution. 1.

$$Dx(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{u}{\sqrt{1-u^2-v^2}} & -\frac{v}{\sqrt{1-u^2-v^2}} \end{pmatrix}$$
$$g_x = Dx^\top Dx = \begin{pmatrix} 1+\frac{u^2}{1-u^2-v^2} & \frac{uv}{1-u^2-v^2} \\ \frac{uv}{1-u^2-v^2} & 1+\frac{v^2}{1-u^2-v^2} \end{pmatrix}$$

2. On the domain: length$(l_1) = ||(0,0)-(0,0.8)||_2 = 0.8$ is possible but we will define a parametrization of the line for practice. (I have no clue why this is aligned to the right, sorry)

$$c : (0,1) \to \mathbb{R}^2$$
$$t \mapsto t \cdot \begin{pmatrix} 0 \\ 0.8 \end{pmatrix}$$
$$\int_0^1 1 \cdot ||c'(t)||dt = \int_0^1 0.8dt = [0.8 \cdot t]_0^1 = 0.8$$
$$\int_0^1 1 \cdot \sqrt{det(g_x(c(t)))}dt = \int_0^1 \sqrt{det \begin{pmatrix} 1 & 0 \\ 0 & 1+\frac{0.64t^2}{1-0.64t^2} \end{pmatrix}}dt$$
$$\int_0^1 \sqrt{1+\frac{0.64t^2}{1-0.64t^2}}dt = 1.15912$$

3. On C we can calculate the angle α by solving $\cos(\alpha) = \frac{\langle x,y\rangle}{||x||\cdot||y||}$ where x and y are any vectors in the direction of the tangent of l_1, l_2. We smartly choose them to have length 1, so $x = (01)^\top$ and $y = (\sqrt{0.5}\sqrt{0.5})^\top$. Then we get

$$\alpha = \arccos(\langle x,y\rangle) = 0.78 = 45$$

To calculate the same angle on the manifold we need to replace the inner product (and norm) by the inner product wrt. the first fundamental form at the point where we measure the angle. Means instead of $x^\top y$ we take $x^\top g_x y$. Remember you can write the norm as $\sqrt{x^\top x}$. Since our lines intersect at the point $(0,0)$ we calculate the fff there:

$$g_x(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

At $(0,0)$ the fff is the identity which equals the standard inner product and therefore all calculations and the angle are the same. But this would not hold if the lines would intersect at any other point because g_x takes values different from the identity there. (You can easily make up an example by yourself if you are interested, for a non-conformal map g_x can not be diagonal matrix.)

Exercise 3 (3 points)**.** Show that the first fundamental form is invariant to rotation and translation in the coordinate map. Let $x_1, x_2 : \mathbb{R}^2 \to \mathbb{R}^3$ be defined such that $x_2(u) = R \cdot x_1(u) + T$ where $R \in \mathbb{R}^{3\times 3}$ is a rotation matrix and $T \in \mathbb{R}^3$. To show is that:

$$Dx_1(u)^\top Dx_1(u) = Dx_2(u)^\top Dx_2$$

Solution. First notice that the translation vanishes in all partial derivatives inside Dx_2 so we are left with $Dx_2 = RDx_1$. Then in the first fundamental form:

$$g_{x_2} = Dx_2^\top Dx_2 = Dx_1^\top \underbrace{R^\top R}_{=Id} Dx_1 = Dx_1^\top Dx_1 = g_{x_1}$$