Analysis of Three-Dimensional Shapes
F. R. Schmidt, M. Vestner, Z. Lähner
Summer Semester 2017

Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 8

Room: 02.09.023

Wed, 12.07.2017, 14:00-16:00

Submission deadline: Tue, 11.07.2017, 23:59 to laehner@in.tum.de

Mathematics: Stiffness matrix

Let X be a vector space. An *inner product* is a function $f : X \times X \to \mathbb{C}$ with the following properties:

1. $f(x,x) \geq 0 \quad \forall\, x \in X$ and $f(x,x) = 0 \Leftrightarrow x = 0$
2. $f(x,y) = \overline{f(y,x)}$
3. $f(x + \alpha x', y) = f(x,y) + \alpha f(x', y) \quad \forall\, x, x', y \in X, \alpha \in \mathbb{C}$

The standard inner product on $X = \mathbb{C}$ is defined as $\langle x, y \rangle = x^\top \bar{y}$. If $\mathbf{M} \in \mathbb{R}^{n \times n}$ is a symmetric, positive definite matrix the $\mathbf{M}$-inner product is defined by $\langle x, y \rangle_M = x^\top M \bar{y}$.

A linear operator $T : X \to X$ is called *self-adjoint* w.r.t. an inner product f if the following holds:

$$f(Tx, y) = f(x, Ty)$$

An *eigenvector* is an element $0 \neq x \in X$ for which there exists a scalar $\lambda \in \mathbb{C}$ such that

$$Tx = \lambda x$$

The scalar λ is called *eigenvalue*.

Exercise 1. Let $\mathbf{L} = \mathbf{M}^{-1}\mathbf{S} \in \mathbb{R}^{n \times n}$ be self-adjoint w.r.t. the $\mathbf{M}$ inner product. Show that the following statements hold.

1. $\mathbf{S}$ is symmetric (self-adjoint) w.r.t. to the standard inner product.
2. The eigenvalues of $\mathbf{L}$ are real.
3. The eigenvectors v_i, v_j with respective eigenvalues $\lambda_i \neq \lambda_j$ are $\mathbf{M}$-orthogonal.
4. $\mathbf{v}_1, ..., \mathbf{v}_k$ are eigenvectors of $\mathbf{L}$ with the same eigenvalue λ, then $\sum_i \alpha_i \mathbf{v}_i$ is also an eigenvector with eigenvalue λ.

Solution. 1.

$$\begin{aligned}\langle Sx, y\rangle &= x^\top S^\top \bar{y} = x^\top S^\top M^{-1} M y \\ &= \langle M^{-1} S x, y\rangle_M = \langle Lx, y\rangle_M \\ &= \langle x, Ly\rangle_M = x^\top M M^{-1} S \bar{y} \\ &= \langle x, Sy\rangle\end{aligned}$$

2. Let λ be an eigenvalue of L such that $Lv = \lambda v$.

$$\begin{aligned}\lambda\langle v, v\rangle_M &= \langle \lambda v, v\rangle_M \\ &= \langle Lv, v\rangle_M = \langle v, Lv\rangle \\ &= \langle v, \lambda v\rangle \\ &= \bar{\lambda}\langle v, v\rangle\end{aligned}$$

3.

$$\begin{aligned}\lambda_i\langle v_i, v_j\rangle &= \langle \lambda_i v_i, v_j\rangle \\ &= \langle L v_i, v_j\rangle = \langle v_i, L v_j\rangle \\ &= \langle v_i, \lambda_j v_j\rangle \\ &= \lambda_j\langle v_i, v_j\rangle\end{aligned}$$

Since we assumed $\lambda_i \neq \lambda_j$ this is only possible when $\langle v_i, v_j\rangle = 0$.

4.

$$\begin{aligned}L\left(\sum_i \alpha_i v_i\right) &= \sum_i \alpha_i L v_i \\ &= \sum_i \alpha_i \lambda v_i \\ &= \lambda\left(\sum_i \alpha_i v_i\right)\end{aligned}$$

Exercise 2. Show that the diagonal entries $\mathbf{S}_{ii} = \int_{\mathcal{M}} \langle \nabla\psi_i, \nabla\psi_i\rangle$ of the stiffness matrix satisfy:

$$\mathbf{S}_{ii} = \sum_{(i,j) \text{ edge at } i} \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{2} = -\sum_j \mathbf{S}_{ij}$$

Hints

- Consider each traingle independently
- $\int_T 1 dp = \int_{T_{ref}} \sqrt{\det g} du = \text{area}(T)$.

- The area of a triangle can be calculated as the half of the product of an edge length and the corresponding height of the triangle.

Solution. Let $\tilde{\phi}_i(p) = \phi_i(x_k(p))$ and $e_1 = x_k(u_1)$, $e_2 = x_k(u_2)$.

For completeness this solution also contains the entries S_{ij} although only $i = j$ was asked for. Feel free to ignore the first part.

First case, $i \neq j$:

$$\begin{aligned} C_{ij} &= \int_S \langle \nabla\phi_i(x), \nabla\phi_j(x) \rangle \mathrm{d}x \\ &= \sum_{k \in \mathcal{T}} \int_{T_k} \langle \nabla\phi_i(p), \nabla\phi_j(p) \rangle \mathrm{d}p \end{aligned}$$

$$\begin{aligned} \int_{T_k} \langle \nabla\phi_i(p), \nabla\phi_j(p) \rangle \mathrm{d}p &= \int_{T_{\text{ref}}} \langle g_k^{-1} \nabla\tilde{\phi}_i(p), g_k^{-1} \nabla\tilde{\phi}_j(p) \rangle \sqrt{det(g_k)} \mathrm{d}p \\ &= \sqrt{det(g_k)} \begin{pmatrix} 1 & 0 \end{pmatrix} g_k^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \underbrace{\int_{T_{\text{ref}}} 1 \ \mathrm{d}p}_{=1/2} \\ &= \frac{1}{2} \frac{\sqrt{det(g_k)}}{det(g_k)} g_k^{12} \\ &= \frac{1}{2} \frac{-\langle e_1, e_2 \rangle}{\sqrt{det(g_k)}} \\ &= -\frac{1}{2} \frac{||e_1|| \cdot ||e_2|| \cdot \cos(\alpha_{ij})}{2 \cdot \text{area}(T_k)} \\ &= -\frac{1}{2} \frac{||e_1|| \cdot ||e_2|| \cdot \cos(\alpha_{ij})}{||e_1|| \cdot ||e_2|| \cdot \sin(\alpha_{ij})} \\ &= -\frac{\cot(\alpha_{ij})}{2} \end{aligned}$$

The summands are only non-zero at triangles adjacent to both i and j. See the lecture slides for a sketch of α_{ij}, β_{ij}.

$$\begin{aligned} C_{ij} &= \sum_{k \in \mathcal{N}_i \cap \mathcal{N}_j} -\frac{\cot \alpha_{ij}}{2} \\ &= -\frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} \end{aligned}$$

Second case, $i = j$:

$$\begin{aligned} C_{ii} &= \int_S \langle \nabla\phi_i(x), \nabla\phi_i(x) \rangle \mathrm{d}x \\ &= \sum_{k \in \mathcal{T}} \int_{T_k} ||\nabla\phi_i(p)||^2 \mathrm{d}p \end{aligned}$$

$$
\begin{aligned}
\int_{T_k} ||\nabla \phi_i(p)||^2 \mathrm{d}p &= \int_{T_{\text{ref}}} ||g_k^{-1} \nabla \tilde{\phi}_i(p)||^2 \sqrt{det(g_k)}\ \mathrm{d}p \\
&= \sqrt{det(g_k)} \begin{pmatrix} 1 & 0 \end{pmatrix} g_k^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \int_{T_{\text{ref}}} 1\ \mathrm{d}p \\
&= \frac{||e_1||^2}{4 \cdot \text{area}(T_k)} \\
&= \frac{w^2}{2 \cdot w \cdot h} \\
&= \frac{w}{2 \cdot h} \\
&= \frac{1}{2}(\cot(\alpha) + \cot(\beta))
\end{aligned}
$$

e_1 is the edge opposing vertex i in each triangle. We can write the diagonal entries as sums over triangles or one can see that the angles showing up are exactly the same as in the entries C_{ij} but paired up differently (see below).

$$
\begin{aligned}
C_{ii} &= \sum_{k \in \mathcal{N}_i} \frac{\cot \alpha_k + \cot \beta_k}{2} \\
&= \sum_{(i,j) \text{ edge at } i} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} = \sum_j C_{ij}
\end{aligned}
$$

Programming: Stiffness matrix

Exercise 3. Download the supplementary material from the homepage. It contains four files describing two 3D triangular meshes.

1. Implement a function `stiffness_matrix.m` that takes a triangle mesh and returns a `sparse` stiffness matrix.
2. Use the `eigs` command to get the first four (ordered by magnitude of the eigenvalue, from small to big) solutions of the generalized eigenvalue problem

$$\lambda \mathbf{M} \phi_i = -\mathbf{S} \phi_i$$

   and visualize them

   - as color coded functions on the shapes.
   - as embeddings of the shapes in $\mathbb{R}^3$ (do not use the first one. Why?).

Hints

- Recap the construction of the mass matrix from sheet 4.

- For each triangle calculate the cot of all three angles and add them to the corresponding positions in the stiffness matrix.
- the `sparse` command automatically adds values if an entry is assigned multiple times.