Chapter 1
Convex Analysis

Convex Optimization for Machine Learning & Computer Vision
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Convex Set

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Convex Optimization

Notations

- $\mathbb{E}$ is a Euclidean space (finite dimensional vector space), equipped with the inner product $\langle \cdot, \cdot \rangle$, e.g. $\langle u, v \rangle = u^\top v$.
- C is a closed, convex subset of $\mathbb{E}$.
- J is a convex objective function.

Convex optimization

$$\text{minimize } J(u) \quad \text{over } u \in C.$$

First questions:

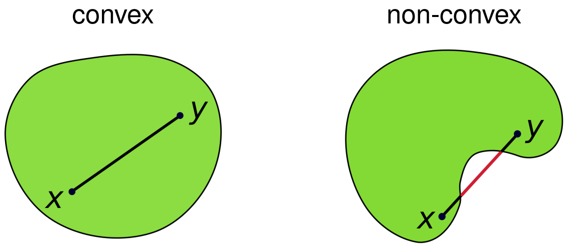
- What is a convex set?
- What is a convex function?

Convex set

Definition

A set C is said to be **convex** if

$$\alpha u + (1-\alpha) v \in C, \quad \forall u, v \in C, \; \forall \alpha \in [0,1].$$



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Recall basic concepts in analysis

Definition

- A set $C \subset \mathbb{E}$ is **open** if $\forall u \in C$, $\exists \epsilon > 0$ s.t. $B_\epsilon(u) \subset C$, where $B_\epsilon(u) := \{v \in \mathbb{E} : \|v - u\| < \epsilon\}$.
- A set $C \subset \mathbb{E}$ is **closed** if its complement $\mathbb{E} \backslash C$ is open.
- The **closure** of a set $C \subset \mathbb{E}$ is

$$\text{cl}\, C = \{u \in \mathbb{E} : \exists \{u^k\} \subset C \text{ s.t. } \lim_{k \to \infty} u^k = u\}.$$

- The **interior** of a set $C \subset \mathbb{E}$ is

$$\text{int}\, C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

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- The **interior** of a set $C \subset \mathbb{E}$ is

$$\operatorname{int} C = \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \subset C\}.$$

- The **relative interior** of a set $C \subset \mathbb{E}$ is

$$\begin{aligned} \operatorname{rint} C &:= \{u \in C : \exists \epsilon > 0 \text{ s.t. } B_\epsilon(u) \cap \operatorname{aff} C \subset C\} \\ &= \{u \in C : \forall v \in C, \exists \alpha > 1 \text{ s.t. } v + \alpha(u - v) \in C\} \end{aligned}$$

  if C is convex. Here aff C stands for the **affine hull** of C.

Basic properties

The following operations preserve the convexity:

- Intersection: $C_1 \cap C_2$
- Summation: $C_1 + C_2 := \{u^1 + u^2 : u^1 \in C_1, u^2 \in C_2\}$
- Closure: $\operatorname{cl} C$
- Interior: $\operatorname{int} C$

– The union of convex sets is not convex in general.

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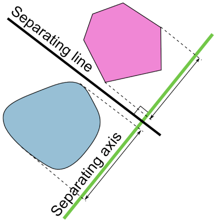
– The union of convex sets is not convex in general.

– *Polyhedral sets* are always convex; *cones* are not necessarily convex.

Convex cone

C is a **cone** if $C = \alpha C$ for any $\alpha > 0$. C is a **convex cone** if C is a cone and is convex as well.

Separation of convex sets



Source: Wikipedia.

Theorem (separation of convex sets)

Let C_1, C_2 be nonempty convex subsets in $\mathbb{E}$ s.t. $C_1 \cap C_2 = \emptyset$ and C_1 is open. Then there exists a hyperplane separating C_1 and C_2, i.e. $\exists v \in \mathbb{E},\ v \neq 0,\ \alpha \in \mathbb{R}$ s.t.

$$\langle v, u^1 \rangle \geq \alpha \geq \langle v, u^2 \rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$

Proof: on board.

Separation of convex sets

Theorem (separation of convex sets)

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$$\left\langle v, u^1 \right\rangle \geq \alpha \geq \left\langle v, u^2 \right\rangle, \quad \forall u^1 \in C_1,\ u^2 \in C_2.$$

Proof: on board.

Remarks

1. The proof works in any Hilbert space.
2. Corollary: In a Hilbert space, any (strongly) closed convex subset C is weakly closed.
3. The above theorem generalizes to any topological vector space, known as the *Hahn-Banach theorem*.

Convex Function

Convex functions

- An **extended real-valued function** J maps from $\mathbb{E}$ to $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$.
- The **domain** of $J : \mathbb{E} \to \overline{\mathbb{R}}$ is

$$\operatorname{dom} J = \{u \in \mathbb{E} : J(u) < \infty\}.$$

- The function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is **proper** if $\operatorname{dom} J \neq \emptyset$.

Definition

We say $J : \mathbb{E} \to \overline{\mathbb{R}}$ is a **convex function** if

1. $\operatorname{dom} J$ is a convex set.
2. For all $u, v \in \operatorname{dom} J$ and $\alpha \in [0, 1]$ it holds that

$$J(\alpha u + (1 - \alpha)v) \leq \alpha J(u) + (1 - \alpha)J(v).$$

We say J is **strictly convex** if the above inequality is strict for all $\alpha \in (0, 1)$ and $u \neq v$.

Examples

- $J_{data}(u) = \|u - f\|_q^q$, where $q \geq 1$ and $\|\cdot\|_q$ is ℓ^q-norm.
- $J_{regu}(u) = \|Ku\|_{q'}^{q'}$, where K is linear transform and $q' \geq 1$.
- $J(u) = J_{data}(u) + \alpha J_{regu}(u)$.

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- $J(u) = J_{data}(u) + \alpha J_{regu}(u)$.
- (Binary) entropy: $J_\epsilon(u) = \epsilon\big(u \log(u) + (1-u)\log(1-u)\big)$.
- Soft max: $J_\epsilon^*(v) = \epsilon \log(1 + \exp(v/\epsilon)) \approx \max(v, 0)$.

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- Soft max: $J_\epsilon^*(v) = \epsilon \log(1 + \exp(v/\epsilon)) \approx \max(v, 0)$.
- **Indicator function** ($C \subset \mathbb{E}$ is closed and convex):

$$\delta_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ \infty & \text{otherwise.} \end{cases}$$

- Formulate constrained optimization with indicator function:

$$\min J(u) \text{ over } u \in C. \quad \Leftrightarrow \quad \min J(u) + \delta_C(u) \text{ over } u \in \mathbb{E}.$$

Basic facts

(As exercises)

- Any norm (over a normed vector space) is a convex function.
- J is a convex function and K is a linear transform $\Rightarrow J(K\cdot)$ is convex function.
- (Jensen's inequality) $J : \mathbb{E} \to \overline{\mathbb{R}}$ is convex iff

$$J\left(\sum_{i=1}^{n} \alpha_i u^i\right) \leq \sum_{i=1}^{n} \alpha_i J(u^i),$$

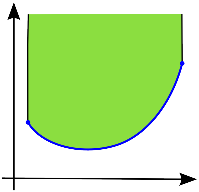
  whenever $\{u^i\}_{i=1}^n \subset \mathbb{E}$, $\{\alpha_i\}_{i=1}^n \subset [0,1]$, $\sum_{i=1}^n \alpha_i = 1$.

Epigraph

Definition

The **epigraph** of a proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is

$$\operatorname{epi} J = \{(u, \alpha) \in \mathbb{E} \times \mathbb{R} : J(u) \leq \alpha\}.$$



epigraph

Theorem

A proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is convex (resp. strictly convex) iff epi J is a convex (resp. strictly convex) set.

Proof: as exercise.

Lipschitz continuity

Definition

Assume $J : \mathbb{E} \to \overline{\mathbb{R}}$ with $\operatorname{rint}\operatorname{dom} J \neq \emptyset$. We say J is **locally Lipschitz** at $u \in \operatorname{rint}\operatorname{dom} J$ with modulus $L_u > 0$ if there exists $\epsilon > 0$ s.t.

$$|J(u^1) - J(u^2)| \leq L_u \|u^1 - u^2\| \quad \forall u^1, u^2 \in B_\epsilon(u) \cap \operatorname{rint}\operatorname{dom} J.$$

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Theorem

A proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is locally Lipschitz at any $u \in \operatorname{rint} \operatorname{dom} J$.

Proof: on board.

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Existence of Minimizer

Global vs. Local minimizer

Recall the optimization of $J : \mathbb{E} \to \overline{\mathbb{R}}$:

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E}.$$

Definition

1. $u^* \in \mathbb{E}$ is a **global minimizer** if $J(u^*) \leq J(u)$ for all $u \in \mathbb{E}$.
2. u^* is a **local minimizer** if $\exists \epsilon > 0$ s.t. $J(u^*) \leq J(u)$ for all $u \in B_\epsilon(u^*)$.
3. In the above definitions, a global/local minimizer is **strict** if $J(u^*) \leq J(u)$ is replaced by $J(u^*) < J(u)$.

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3. In the above definitions, a global/local minimizer is **strict** if $J(u^*) \leq J(u)$ is replaced by $J(u^*) < J(u)$.

Theorem

For any proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, if $u^* \in \operatorname{dom} J$ is a local minimizer of J, then it is also a global minimizer.

Proof: on board.

Convex Set

Convex Function

Existence of Minimizer

Subdifferential

Convex Conjugate

Duality Theory

Does a minimizer always exist?

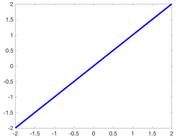
- Consider

$$\text{minimize } J(u) \quad \text{over } u \in \mathbb{E},$$

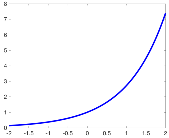
where $J : \mathbb{E} \to \overline{\mathbb{R}}$ is a proper, convex function.

- Some counterexamples for $J : \mathbb{R} \to \overline{\mathbb{R}}$:

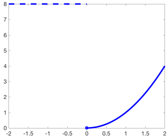
u



$\exp u$



$u^2 + \delta\{u > 0\}$



- We shall formalize our observations and derive sufficient conditions for existence.

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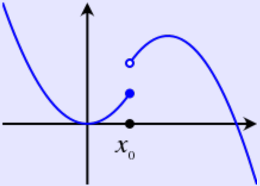
Sufficient conditions for existence

Definition

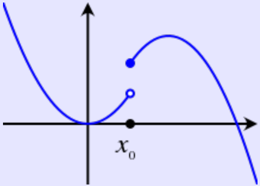
1. J is **bounded from below** if $J(\cdot) \geq C$ for some $C \in \mathbb{R}$.
2. J is **coercive** if $J(u) \to \infty$ whenever $\|u\| \to \infty$.
   - Proposition: J is coercive if dom J is bounded.
3. J is **lower semi-continuous** (lsc) at u^* if

$$J(u^*) \leq \liminf_{u \to u^*} J(u).$$

lsc



not lsc



- Proposition: J is lsc iff epi J is closed.

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Theorem

Any proper function $J : \mathbb{E} \to \overline{\mathbb{R}}$, which is bounded from below, coercive, and lsc (everywhere), has a (global) minimizer.

Proof: on board.

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Remarks for infinite dimensions

1. Weak compactness in reflexive Banach (e.g. Hilbert) sp.
2. J is convex and strongly lsc $\Rightarrow$ J is weakly lsc.

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Uniqueness

- Recall that a function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is strictly convex if

$$J(\alpha u + (1-\alpha)v) < \alpha J(u) + (1-\alpha)J(v),$$

for all $u, v \in \operatorname{dom} J,\ u \neq v,\ \alpha \in (0,1)$.

Theorem

The minimizer of a strictly convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ is unique.

Proof: on board.

Subdifferential

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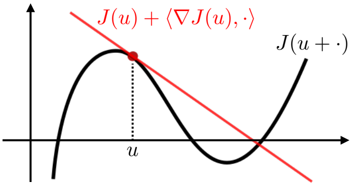
Differential



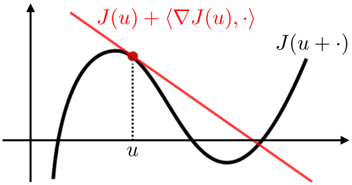
Definition

$J : \mathbb{E} \to \overline{\mathbb{R}}$ is called (Fréchet) **differentiable** at $u \in \operatorname{int} \operatorname{dom} J$ and $\nabla J(u) \in \mathbb{E}$ is the (Fréchet) **differential** of J at u if

$$\lim_{h \to 0} \frac{|J(u+h) - J(u) - \langle \nabla J(u), h \rangle|}{\|h\|} = 0.$$

J is **continuously differentiable** at $u \in \operatorname{int} \operatorname{dom} J$ if $\nabla J(\cdot)$ is continuous on $(\operatorname{dom} J) \cap B_\epsilon(u)$ for some $\epsilon > 0$.

Differential



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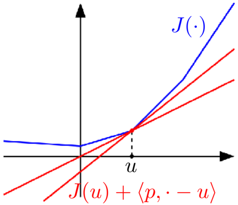
Remark

If $\mathbb{E}$ is a topological vector space, $\nabla J(u)$ is treated as a *dual* object in $\mathbb{E}^*$, and $\langle \nabla J(u), h \rangle_{\mathbb{E}^*, \mathbb{E}}$ as *duality pairing*.

Subdifferential

Now we generalize differentiability from differentiable functions to non-differentiable convex functions.



Definition

The **subdifferential** of a convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$ at $u \in \operatorname{dom} J$ is defined by

$$\partial J(u) = \{p \in \mathbb{E} : J(v) \geq J(u) + \langle p, v - u \rangle \ \ \forall v \in \mathbb{E}\} .$$

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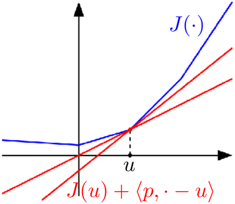

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Geometric interpretation

$p \in \partial J(u)$ iff $(p, -1)$ is a normal vector for the supporting hyperplane of $\operatorname{epi} J$ at $(u, J(u))$.

Examples and basic facts

Basic facts

1. $\partial J(\cdot)$ is a *set-valued* map.
2. If J is cont. differentiable at u, then $\partial J(u) = \{\nabla J(u)\}$.

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Examples

1. $J(u) = \|u\| \;\Rightarrow\; \partial J(0) = \{p : \|p\|_* \leq 1\}$, where $\|\cdot\|_*$ is the **dual norm** of $\|\cdot\|$, i.e., $\|p\|_* = \sup\{\langle p, u\rangle : \|u\| \leq 1\}$.
2. Given any closed, convex subset $C \subset \mathbb{E}$ and $u \in C$,

$$\partial \delta_C(u) = \{p \in \mathbb{E} : \langle p, v - u\rangle \leq 0 \;\forall v \in C\} =: N_C(u),$$

known as the **normal cone** of C at u.

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   known as the **normal cone** of C at u.
3. (Exercise) $X \in \mathbb{R}^{m\times n} \mapsto \|X\|_{1,2} = \sum_{i=1}^m \left(\sum_{j=1}^n |X_{i,j}|^2\right)^{1/2}$.
4. (Exercise) $X \in \mathbb{R}^{m\times n} \mapsto \|X\|_{nuclear} = \sum_i \sigma_i(X)$, i.e., sum of singular values.

Subdifferential calculus

Theorem (chain rule under linear transform)

Let $\widetilde{J}(\cdot) = J(K\cdot)$ with some convex function J and linear transform K. Then

$$\partial\widetilde{J}(u) = K^\top \partial J(Ku)$$

whenever $Ku \in \operatorname{rint}\operatorname{dom} J$.

Example: $J(u) = \|Ku\| \Rightarrow \partial J(u) = K^\top \partial\|\cdot\|(Ku)$.

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Theorem (summation rule)

If $\widetilde{J}(\cdot) = J_1(\cdot) + J_2(\cdot)$ for some convex functions J_1 and J_2, then

$$\partial\widetilde{J}(u) = \partial J_1(u) + \partial J_2(u)$$

for any $u \in \operatorname{rint}\operatorname{dom} J_1 \cap \operatorname{rint}\operatorname{dom} J_2$.

Warning: not true if J_1 or J_2 is non-convex, e.g. $0 = |\cdot| + (-|\cdot|)$.

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Properties of subdifferential map

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a convex function. Then ∂J is a **monotone operator**, i.e. $\forall u^1, u^2 \in \operatorname{dom} J,\ p^1 \in \partial J(u^1),\ p^2 \in \partial J(u^2)$:

$$\langle p^1 - p^2, u^1 - u^2 \rangle \geq 0.$$

Proof: on board.

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Proof: on board.

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a convex function. Then for any $u \in \operatorname{int} \operatorname{dom} J$, $\partial J(u)$ is a nonempty, compact, and convex subset.

Proof: on board.

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Proof: on board.

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a convex function. Then for any $u \in \operatorname{int} \operatorname{dom} J$, $\partial J(u)$ is a nonempty, compact, and convex subset.

Proof: on board.

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$ be a proper, convex, lsc function. Then the set-valued map $\partial J(\cdot)$ is **closed**, i.e. $p^* \in \partial J(u^*)$ whenever

$$\exists (u^k, p^k) \to (u^*, p^*) \in (\operatorname{dom} J) \times \mathbb{E} \ \text{ s.t. } p^k \in \partial J(u^k) \ \ \forall k.$$

Proof: on board.

Optimality condition

Theorem

Given any proper convex function $J : \mathbb{E} \to \overline{\mathbb{R}}$, the sufficient and necessary condition for u^* being a (global) minimizer for J is

$$0 \in \partial J(u^*).$$

Proof: on board.

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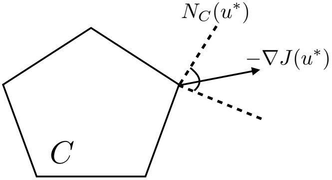
Constrained optimization as special case

If u^* minimizes $\widetilde{J} = J + \delta_C$ with convex function $J : \mathbb{E} \to \mathbb{R}$ and closed convex subset $C \subset \mathbb{E}$, then $0 \in \partial \widetilde{J}(u^*) \Leftrightarrow$

$$0 \in \partial J(u^*) + N_C(u^*).$$



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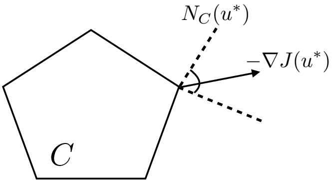

Remark

The optimality condition $0 \in \partial J(u^*) + N_C(u^*)$ is *geometric*. More explicit characterization replies on the *algebraic* representation of $N_C(u^*)$, e.g., the Karush-Kuhn-Tucker (KKT) conditions, typically under certain *constraint qualifications*.

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Example: Linear-inequality constraints

Let $C = \{u \in \mathbb{R}^n : Au \leq b\}$ where $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$ has linearly independent rows. Then

$$N_C(u) = \{A^\top \lambda : \lambda \geq 0, \lambda_i = 0 \text{ if } (Au - b)_i < 0\}.$$

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Convex Conjugate

Convex conjugate

Definition

Given a function $J : \mathbb{E} \to \overline{\mathbb{R}}$, the **convex conjugate** (a.k.a. Legendre-Fenchel transform) of J is defined by

$$J^*(p) = \sup_{u \in \mathbb{E}} \{\langle u, p \rangle - J(u)\} \quad \forall p \in \mathbb{E}.$$

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Examples (as exercise)

1. $J(u) = \langle w, u \rangle \;\Rightarrow\; J^*(p) = \delta\{p = w\}$. ($w \in \mathbb{E}$ given)
2. $J(u) = \|u\| \;\Rightarrow\; J^*(p) = \delta\{\|p\|_* \leq 1\}$. ($\|\cdot\|_*$ is the *dual norm* of $\|\cdot\|$, i.e. $\|p\|_* = \sup\{\langle p, u \rangle : \|u\| \leq 1\}$)
3. $J(u) = \frac{1}{q}\|u\|_q^q \;\Rightarrow\; J^*(p) = \frac{1}{q'}\|p\|_{q'}^{q'}$. ($q \in [1, \infty]$, $\frac{1}{q} + \frac{1}{q'} = 1$)
4. $J(u) = \sum_i u_i \log u_i + \delta_{\Delta^{n-1}}(u) \Rightarrow J^*(p) = \log(\sum_i \exp(p_i))$.

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Basic facts (as exercise)

- Scalar multiplication: $\widetilde{J}(\cdot) = \alpha J(\cdot) \;\Rightarrow\; \widetilde{J}^*(\cdot) = \alpha J^*(\cdot/\alpha)$.
- Translation: $\widetilde{J}(\cdot) = J(\cdot - z) \;\Rightarrow\; \widetilde{J}^*(\cdot) = J^*(\cdot) + \langle \cdot, z \rangle$.

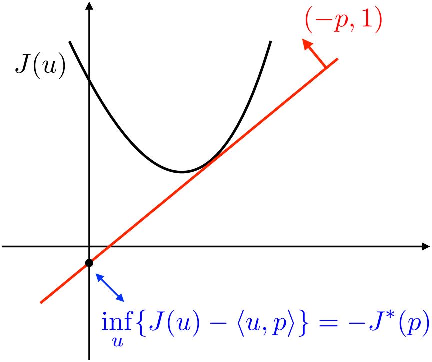
Geometric interpretation

Geometrically, convex conjugation maps

the normal vector of a supporting hyperplane to the epigraph

to

the intersection with the vertical axis.



Fenchel-Young inequality, order reversing property

Theorem (Fenchel-Young inequality)

For any $J : \mathbb{E} \to \overline{\mathbb{R}}$ and $(u, p) \in \mathbb{E} \times \mathbb{E}$, we have

$$J(u) + J^*(p) \geq \langle u, p \rangle .$$

The equality holds iff $p \in \partial J(u)$ for $(u, p) \in \operatorname{dom} J \times \operatorname{dom} J^*$.

Proof: (i) $J(u) + J^*(p) \geq \langle u, p \rangle$ follows directly from the definition of convex conjugate.

(ii) The equality holds only if $(u, p) \in \operatorname{dom} J \times \operatorname{dom} J^*$. Moreover, $p \in \partial J(u)$ is the sufficient and necessary condition for $\min_{u \in \mathbb{E}} \{J(u) - \langle u, p \rangle\}$.

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Theorem (order reversing)

For any $J_1, J_2 : \mathbb{E} \to \overline{\mathbb{R}}$, we have $J_1^* \leq J_2^*$ whenever $J_1 \geq J_2$.

Proof: For any (u, p), we have $\langle u, p \rangle - J_1(u) \leq \langle u, p \rangle - J_2(u)$. Taking supremum over u on both sides yields $J_1^*(p) \leq J_2^*(p)$.

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Biconjugate

Theorem

Let $J : \mathbb{E} \to \overline{\mathbb{R}}$, and $J^{**} = (J^*)^*$ is the **biconjugate** of J.

In general:

1. $J^{**}(\cdot) \leq J(\cdot)$.
2. J^* is convex and lsc.

If J is proper, convex, and lsc, then:

3. $J^{**}(\cdot) = J(\cdot)$.
4. $p \in \partial J(u)$ iff $u \in \partial J^*(p)$.

Proof: on board.

Regularity of J and J^*

Definition

1. $J : \mathbb{E} \to \mathbb{R}$ is μ-**strongly convex** if $\exists \mu > 0$ s.t. $J(\cdot) - \frac{\mu}{2}\|\cdot\|^2$ is convex.
2. $J : \mathbb{E} \to \mathbb{R}$ is L-**Lipschitz differentiable** (a.k.a. L-smooth) if J is differentiable and ∇J is Lipschitz with modulus L.

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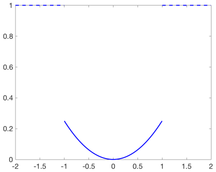
Theorem

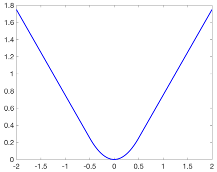
Assume that $J : \mathbb{E} \to \overline{\mathbb{R}}$ is proper, convex, and lsc. Then J is μ-strongly convex iff J^* is $\frac{1}{\mu}$-Lipschitz differentiable.

Proof: on board.

truncated quadratic



Huber function



Duality Theory

Fenchel-Rockafellar duality

- Consider

$$\inf_{u \in \mathbb{R}^n} \{F(Ku) + G(u)\},$$

where $K \in \mathbb{R}^{m \times n}$, and F, G are proper, convex, and lsc.

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- The **weak duality** always holds:
$$\begin{aligned}\mathcal{P}^* &:= \inf_u\{F(Ku)+G(u)\}\\ &= \inf_u\sup_p\{\langle p,Ku\rangle - F^*(p)+G(u)\}\\ &\geq \sup_p\inf_u\{\left\langle K^\top p,u\right\rangle + G(u)-F^*(p)\}\\ &= \sup_p\{-G^*(-K^\top p)-F^*(p)\} =: \mathcal{D}^*.\end{aligned}$$

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&= \sup_p \{-G^*(-K^\top p) - F^*(p)\} =: \mathcal{D}^*.
\end{aligned}$$

- Define the **duality gap**:

$$\mathcal{G}(u, p) = F(Ku) + G(u) + G^*(-K^\top p) + F^*(p).$$

Note that $\mathcal{G}(u, p) = 0$ is an optimality criterion.

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Fenchel-Rockafellar duality

- $\mathcal{G}(u^*, p^*) = 0 \Leftrightarrow \mathcal{P}^* = \mathcal{D}^* \Leftrightarrow (u^*, p^*)$ solves the **saddle point problem** with $\mathcal{L}(u, p) := \langle p, Ku \rangle - F^*(p) + G(u)$:

$$\mathcal{L}(u^*, p) \leq \mathcal{L}(u^*, p^*) \leq \mathcal{L}(u, p^*) \quad \forall (u, p).$$

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Theorem (Fenchel-Rockafellar duality)

Assume $\exists \bar{u} \in \operatorname{dom} G$ s.t. F is continuous at $K\bar{u}$. Then the **strong duality** holds: $\mathcal{P}^* = \mathcal{D}^*$. Moreover, (u^*, p^*) is the optimal solution pair iff

$$\begin{cases} Ku^* \in \partial F^*(p^*), \\ -K^\top p^* \in \partial G(u^*). \end{cases}$$

Proof: on board.