Convex Optimization for Machine Learning and Computer Vision

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Weekly Exercises 1

Room: 02.09.023
Wednesday, 25.04.2018, 12:15-14:00
Submission deadline: Monday, 23.04.2018, 16:15, Room 02.09.023

Theory: Convex Sets (12+8 Points)

Exercise 1 (4 Points)**.** Let $\mathcal{C}$ be a family of convex sets in $\mathbb{R}^n$, $C_1, C_2 \in \mathcal{C}$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$. Prove convexity of the following sets:

- $\bigcap_{C\in\mathcal{C}} C$
- $P := \{x \in \mathbb{R}^n : Ax \leq b\}$
- $C_1 + C_2 := \{x + y : x \in C_1, y \in C_2\}$ (the Minkowski sum of C_1 and C_2)
- $\lambda C_1 := \{\lambda x : x \in C_1\}$ (the λ-dilatation of C_1).

Exercise 2 (4 Points)**.** Let $\emptyset \neq X \subset \mathbb{R}^n$. Prove the equivalence of the following statements:

- X is closed.
- Every convergent sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ attains its limit in X.

Exercise 3 (4 Points)**.** Prove that if the set $C \subset \mathbb{R}^n$ is convex, then $\sum_{i=1}^N \lambda_i x_i \in C$ with $x_1, x_2, \dots, x_N \in C$ and $0 \leq \lambda_1, \lambda_2, \dots, \lambda_N \in \mathbb{R}$, $\sum_{i=1}^N \lambda_i = 1$.

Hint: Use induction to prove.

Definition (Convex Hull)**.** The convex hull $\operatorname{conv}(S)$ of a finite set of points $S \subset \mathbb{R}^n$ is defined as

$$\operatorname{conv}(S) := \left\{ \sum_{i=1}^{|S|} a_i x_i : x_i \in S, \sum_{i=1}^{|S|} a_i = 1, a_i \geq 0 \right\}$$

Exercise 4 (8 Points)**.** Prove the following statement: Let $n \in \mathbb{N}$ and let $A \subset \mathbb{R}^n$ contain $n + 2$ elements: $|A| = n + 2$. Then there exists a partition of A into two disjoint sets A_1, A_2

$$A = A_1 \dot{\cup} A_2,$$

(meaning that $A_1 \cap A_2 = \emptyset$) so that the convex hulls of A_1 and A_2 intersect:

$$\mathrm{conv}(A_1) \cap \mathrm{conv}(A_2) \neq \emptyset.$$

You may use the following hint. Don't forget to prove the hint!

Hint: Let $x_1, \ldots, x_{n+2} \in \mathbb{R}^n$. Then the set $\{x_1 - x_{n+2}, \ldots, x_{n+1} - x_{n+2}\}$ is linearly dependent and there exist multipliers $a_1, \ldots, a_{n+2}$, not all of which are zero, so that

$$\sum_{i=1}^{n+2} a_i x_i = 0, \quad \sum_{i=1}^{n+2} a_i = 0.$$

The desired partition is formed via all points corresponding with $a_i \geq 0$ and all points with $a_i < 0$.