Convex Optimization for Machine Learning and Computer Vision

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Weekly Exercises 2

Room: 02.09.023

Wednesday, 02.05.2018, 12:15-14:00

Submission deadline: Monday, 30.04.2018, 16:15, Room 02.09.023

Convex sets and functions (12 Points + 4 Bonus)

Exercise 1 (4 Points)**.** Let $J : \mathbb{E} \to \bar{\mathbb{R}}$ be proper. Prove the equivalence of the following statements:

- J is convex.
- $\operatorname{epi}(J) := \left\{ \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathbb{E} \times \mathbb{R} : J(u) \leq \alpha \right\}$ is convex.

Exercise 2 (4 Points)**.** Show that the following functions $J : \mathbb{R}^n \to \overline{\mathbb{R}}$ are convex:

- $J(u) = \|u\|$, for any norm $\|\cdot\|$ over a normed vector space.
- $J(u) = F(Ku)$, for convex $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ and linear $K : \mathbb{R}^m \to \mathbb{R}^n$.
- (Jensen's inequality) J is convex iff

$$J\left(\sum_{i=1}^{n} \alpha_i u^i\right) \leq \sum_{i=1}^{n} \alpha_i J(u^i),$$

  whenever $\{u^i\}_{i=1}^n \subset \mathbb{R}^n$, $\{\alpha_i\}_{i=1}^n \subset [0,1]$, $\sum_{i=1}^n \alpha_i = 1$.

Exercise 3 (4 Points)**.** Let $U \subset \mathbb{E}$ open and convex and let $J : U \to \mathbb{R}$ be twice continuously differentiable. Prove the equivalence of the following statements:

- J is convex.
- For all $u \in U$ the Hessian $\nabla^2 J(u)$ is positive semidefinite ($\forall\, v \in \mathbb{E} : v^\top \nabla^2 J(u) v \geq 0$).

Hints: You can use that for $u, v \in U$ it holds that J is convex iff

$$(v - u)^\top \nabla J(u) \leq J(v) - J(u).$$

Further recall that there are two variants of the Taylor expansion:

$$J(u + td) = J(u) + td^\top \nabla J(u) + \frac{t^2}{2} d^\top \nabla^2 J(u) d + o(t^2)$$

with $\lim_{t\to 0} \frac{o(t^2)}{t^2} = 0$ and

$$J(u+d) = J(u) + d^\top \nabla J(u) + \frac{1}{2} d^\top \nabla^2 J(u+td) d$$

for appropriate $t \in (0,1)$.

Exercise 4 (4 points)**.** Prove the following statement using induction over m: Let $K_1, \ldots, K_m \subset \mathbb{R}^n$, $m \geq n+1$, be convex, such that for all $\mathcal{I} \subset \{1, \ldots, m\}$ with $|\mathcal{I}| = n+1$ it holds that $\bigcap_{i\in\mathcal{I}} K_i \neq \emptyset$. Then $\bigcap_{i=1}^m K_i \neq \emptyset$.

Hint: Use exercise 4 from the first exercise sheet.

Programming: Inpainting(Due date: 07.05) (12 Points)

Exercise 5 (12 Points)**.** Write a MATLAB program that solves the inpainting problem for the vegetable image:

$$\min_{u\in\mathbb{R}^{n\times m}} \sum_{i,j} (u_{i,j} - u_{i-1,j})^2 + (u_{i,j} - u_{i,j-1})^2 \quad \text{s.t. } u_{i,j} = f_{i,j} \ \forall (i,j) \in I,$$

with index set I of pixels to keep. Those can be identified as the white pixels of the mask image.
Hint: The constrained optimization problem can be reformulated so that it becomes unconstrained: Rewrite the objective as a least squares problem in terms of the unknown intensities $u_{i,j}$, $(i,j) \notin I$ using sparse linear operators: Find linear operators X, Y s.t. u can be decomposed as

$$u = X\tilde{u} + Yf$$

where $\tilde{u}$ contains only the unknown intensities. Optimize for $\tilde{u}$ instead of u. You may use MATALBs `mldivide`.