

**Machine Learning for Computer Vision**  
**Winter term 2017**

20. April 2018  
Topic: Linear Algebra

**Exercise 1: Warm up**

- a) What multiple of  $a = (1, 1, 1)$  is closest to the point  $b = (2, 4, 4)$ ? Find also the closest point to  $a$  on the line through  $b$ .

There is some vector  $p = \lambda a, \lambda \neq 0$  which is closest to  $b$ . Then  $p$  is perpendicular to the vector  $b - p$  which means  $p^T(b - p) = 0$ . We just need to find  $\lambda$ , so we solve  $\lambda a^T(b - \lambda a) = 0$  and get  $\lambda = \frac{a^T b}{a^T a}$ .

Plugging in the numbers, we get  $\lambda = \frac{10}{3}$ , so the closest point is  $\lambda a = \frac{10}{3}(1, 1, 1)$ . Equivalently the closest point to  $a$  is  $\mu b = \frac{10}{36}b = \frac{10}{36}(2, 4, 4)$ .

- b) Prove that the trace of  $P = aa^T/a^T a$  always equals 1.

$$\text{We just unfold } aa^T = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} (a_1 \dots a_n) = \begin{bmatrix} a_1^2 & a_1 a_2 & \dots & a_1 a_n \\ a_2 a_1 & a_2^2 & \dots & a_2 a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & \dots & a_n^2 \end{bmatrix}.$$

Also  $a^T a = \sum_i a_i^2$ . Therefore the trace of  $P$  is  $\text{Tr}(P) = \text{Tr}(aa^T/a^T a) = \frac{a_1^2 + \dots + a_n^2}{\sum_i a_i^2} = 1$ .

- c) Show that the length of  $Ax$  equals the length of  $A^T x$  if  $AA^T = A^T A$ .

$$\|Ax\|^2 = (Ax)^T(Ax) = x^T A^T A x = x^T A A^T x = (A^T x)^T(A^T x) = \|A^T x\|^2.$$

- d) Which  $2 \times 2$  matrix projects the x,y plane onto the line  $x + y = 0$ ?

We are looking for the matrix  $A \in \mathbb{R}^{2 \times 2}$  that when multiplied with any vector  $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$  gives us a vector  $u$  that is a *projection* of  $v$  on the line  $x + y = 0$  or otherwise it is a vector  $p = \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ . This means that  $Av = p$  and  $p^T(v - p) = 0$ .

Solving for  $\lambda \neq 0$  we get

$$\begin{aligned}
 p^T(v - p) &= 0 \\
 \lambda(1 \quad -1) \left( \begin{pmatrix} x \\ y \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) &= 0 \\
 \lambda(x - y) - 2\lambda^2 &= 0 \\
 \lambda &= \frac{1}{2}(x - y) \\
 \Rightarrow p &= \frac{1}{2}(x - y) \begin{pmatrix} 1 \\ -1 \end{pmatrix}
 \end{aligned}$$

So we have

$$\begin{aligned}
 Av &= p \\
 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{2}(x - y) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
 \Rightarrow \begin{cases} a_{11}x + a_{12}y &= \frac{1}{2}x - \frac{1}{2}y \\ a_{21}x + a_{22}y &= -\frac{1}{2}x + \frac{1}{2}y \end{cases}
 \end{aligned}$$

And since we have no other constraint for  $A$ , we use the obvious solution

$$A = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

## Exercise 2: Determinants

- a) If a square matrix  $A$  has determinant  $\frac{1}{2}$ , find  $\det(2A)$ ,  $\det(-A)$ ,  $\det(A^2)$  and  $\det(A^{-1})$ .

$$\begin{aligned}
 \det(2A) &= 2^n \det(A) = 2^n \frac{1}{2} = 2^{n-1} \\
 \det(-A) &= (-1)^n \det(A) = \pm \frac{1}{2} \\
 \det(A^2) &= \det(AA) = \det(A) \det(A) = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \\
 \det(A^{-1}) &= \det(A)^{-1} = \left(\frac{1}{2}\right)^{-1} = 2
 \end{aligned}$$

- b) Find the determinants of

$$A = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad U^T \text{ and } U^{-1}$$

$$\det(A) = 0 \quad (\text{A has rank 1 so it is not invertible})$$

$$\det(U) = \prod_{\lambda \in \{4, 1, 2, 2\}} \lambda = 16 \quad (\text{product of the eigenvalues which lie on the diagonal on a triangular matrix})$$

$$\det(U^T) = \det(U) = 16$$

$$\det(U^{-1}) = \det(U)^{-1} = \frac{1}{16}$$

### Exercise 3: Eigenvalues and Eigenvectors

a) Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \text{ their traces and their determinants.}$$

$$\det(A - \lambda I) = (3 - \lambda)(1 - \lambda)(-\lambda) = 0 \Rightarrow \lambda \in \{3, 1, 0\}$$

To find the eigenvectors we plug in the eigenvalues and solve the linear system  $Ax = \lambda x$  for  $x \neq 0$ . The corresponding eigenvectors are then

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

The trace and determinant are

$$\begin{aligned} \text{Tr}(A) &= 3 + 1 + 0 = 4 \\ \det(A) &= 0 \end{aligned}$$

For matrix  $B$  we have

$$\begin{aligned} \det(B - \lambda I) &= (-\lambda)(2 - \lambda)(-\lambda) + 2(-2)(2 - \lambda) = 0 \\ &(\lambda^2 - 4)(2 - \lambda) = 0 \\ &(\lambda + 2)(\lambda - 2)(2 - \lambda) = 0 \\ &\Rightarrow \lambda \in \{-2, 2, 2\} \end{aligned}$$

The corresponding eigenvectors are then

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The trace and determinant are

$$\begin{aligned}\text{Tr}(B) &= 0 + 2 + 0 = 2 \\ \det(B) &= 2(0 - 4) = -8\end{aligned}$$

Typically eigenvectors are normalized to have length 1 but any multiple of an eigenvector is also an eigenvector.

- b) Using the characteristic polynomial, find the relationship between the trace, the determinants and the eigenvalues of any square matrix  $A$ .

We can factor the characteristic polynomial as a function of  $\lambda$  as

$$\det(A - \lambda I) = p(\lambda) = (-1)^n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \quad (1)$$

where  $\lambda_i$  are the roots of the polynomial and the eigenvalues of  $A$ . We can simply set  $\lambda = 0$  and find that

$$\begin{aligned}\det(A) &= p(0) = (-1)^n(-\lambda_1) \cdots (-\lambda_n) = (-1)^n \prod_{i=1}^n (-\lambda_i) = (-1)^n \prod_{i=1}^n (-1)(\lambda_i) \\ &= (-1)^n (-1)^n \prod_{i=1}^n \lambda_i = (-1)^{2n} \prod_{i=1}^n \lambda_i = \prod_{i=1}^n \lambda_i\end{aligned}$$

So the determinant of a matrix is equal to the product of its eigenvalues.

Let us deal with the trace. Consider the  $2 \times 2$  case

$$\begin{aligned}\det(A) &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \\ \det(A - \lambda I) &= \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= ad - bc - \lambda(a + d) + \lambda^2 \\ &= \lambda^2 - \lambda \cdot \text{Tr}(A) + \det(A)\end{aligned}$$

Considering the  $n \times n$  case and focusing on the diagonal, we find that

$$\det(A - \lambda I) = (-\lambda)^n + (-\lambda)^{n-1} \cdot \text{Tr}(A) + \sum_{j=1}^{n-2} \beta_j \lambda^j + \det(A) \quad (2)$$

Comparing equations (1) and (2) we see that

$$\text{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{i=1}^n \lambda_i \quad (3)$$

- c) Diagonalize the unitary matrix  $V$  to reach  $V = U\Lambda U^*$ . All  $|\lambda| = 1$ .  $V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$

We have

$$\begin{aligned} \det(V - \lambda I) &= \left(\frac{1}{\sqrt{3}} - \lambda\right)\left(-\frac{1}{\sqrt{3}} - \lambda\right) - \frac{1}{3}(1+i)(1-i) \\ &= \left(\frac{1}{\sqrt{3}} - \lambda\right)\left(-\frac{1}{\sqrt{3}} - \lambda\right) - \frac{2}{3} \\ &= -\frac{1}{3} + \lambda^2 - \frac{2}{3} \\ &= \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) \end{aligned}$$

Eigenvalues are  $\lambda \in \{1, -1\}$  and corresponding eigenvectors are

$$x_1 = \frac{1}{\sqrt{1+2c^2}} \begin{pmatrix} 1 \\ c+ic \end{pmatrix} \quad \text{and} \quad x_2 = \frac{1}{\sqrt{1+2c^2}} \begin{pmatrix} -c+ic \\ 1 \end{pmatrix}$$

where  $c = \frac{\sqrt{3}-1}{2}$ .

Note that we could arrange the eigenvectors differently but since the matrix  $U$  is unitary, we have to keep the diagonal entries real. Now we can write matrix  $U$  as

$$U = \frac{1}{\sqrt{1+2c^2}} \begin{bmatrix} 1 & -c+ic \\ c+ic & 1 \end{bmatrix}$$

Therefore our decomposition can be written as

$$V = \frac{1}{1+2c^2} \begin{bmatrix} 1 & -c+ic \\ c+ic & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & c-ic \\ -c-ic & 1 \end{bmatrix}$$

- d) Suppose  $T$  is a  $3 \times 3$  upper triangular matrix with entries  $t_{ij}$ . Compare the entries of  $T^*T$  and  $TT^*$ . Show that if they are equal, then  $T$  must be diagonal. (All normal triangular matrices are diagonal)

Let  $T = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$  with  $a, b, c, d, e, f \in \mathbb{C}$ .

Then

$$\begin{aligned} T^*T &= \begin{bmatrix} \bar{a} & 0 & 0 \\ \bar{b} & \bar{d} & 0 \\ \bar{c} & \bar{e} & \bar{f} \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} \bar{a}a & \bar{a}b & \bar{a}c \\ \bar{b}a & \bar{b}b + \bar{d}d & \bar{b}c + \bar{d}e \\ \bar{c}a & \bar{c}b + \bar{e}d & \bar{c}c + \bar{e}e + \bar{f}f \end{bmatrix} \\ TT^* &= \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} \bar{a} & 0 & 0 \\ \bar{b} & \bar{d} & 0 \\ \bar{c} & \bar{e} & \bar{f} \end{bmatrix} = \begin{bmatrix} a\bar{a} + b\bar{b} + c\bar{c} & b\bar{d} + c\bar{e} & c\bar{f} \\ d\bar{b} + e\bar{c} & d\bar{d} + e\bar{e} & e\bar{f} \\ f\bar{c} & f\bar{e} & f\bar{f} \end{bmatrix} \end{aligned}$$

Now if  $TT^* = T^*T$  we see from the diagonal entries that  $-b\bar{b} = c\bar{c}$  and  $\bar{b}b = e\bar{e}$ . So, it must be that  $b = c = e = 0$  and therefore  $T$  is diagonal.

### Exercise 4: Singular Value Decomposition

- a) Find the singular values and singular vectors of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$

Eigenvalues of  $A^T A$  are

$$\det(A^T A - \lambda I) = \lambda(\lambda - 85) = 0 \\ \Rightarrow \lambda \in \{0, 85\}$$

Eigenvectors of  $A^T A$  are  $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$  with norm  $\sqrt{17}$ .

Eigenvalues of  $AA^T$  are also  $\lambda \in \{0, 85\}$

Eigenvectors of  $AA^T$  are  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  with norm  $\sqrt{5}$ .

Therefore:

$$A = \frac{1}{\sqrt{17}} \begin{bmatrix} -4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{85} \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

- b) Explain how  $UDV^T$  expresses  $A$  as a sum of  $r$  rank-1 matrices:  $A = \sigma_1 u_1 v_1^T + \dots + \sigma_r u_r v_r^T$

We see the factorization as

$$A = UDV^T = U(DV^T) = [u_1 \dots u_m] \begin{pmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ & \ddots & & & \\ 0 & & \sigma_r & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & \ddots & 0 \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} \end{pmatrix} \\ = [u_1 \dots u_m] \begin{bmatrix} \sigma_1 v_1^T \\ \vdots \\ \sigma_r v_r^T \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sigma_1 u_1 v_1^T + \dots + \sigma_r u_r v_r^T + 0 \cdot u_{r+1} + \dots + 0 \cdot u_m$$

Note that for the rank it holds  $r \leq m$  and  $r \leq n$ .

- c) If  $A$  changes to  $4A$  what is the change in the SVD?

If  $A = UDV^*$  then  $4A = 4UDV^* = U(4D)V^*$ . We apply the scaling to the singular values and leave the singular vectors normalized as they are.

What is the SVD for  $A^T$  and for  $A^{-1}$  ?

If  $A = UDV^*$  then  $A^T = (UDV^*)^T = VD^T U^T$  The singular values stay in the diagonal, but the dimensions of matrix  $D$  swap.

If  $A = UDV^*$  then we can only compute the pseudoinverse  $A^+ = (UDV^*)^+ = (V^*)^{-1} D^+ U^{-1} = VD^+ U^*$  Since  $U, V$  are unitary, their (conjugate) transpose is also their inverse. The reciprocals of the singular values are in the diagonal.

d) Find the SVD and the pseudoinverse of  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$  ,  $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$   
and  $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

The SVD of A will be  $A = UDV^*$  where  $U$  is  $1 \times 1$  meaning a scalar and since it is unitary it is 1, therefore  $A = DV^*$ .

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad AA^T = 4$$

Then

$$\det(AA^T - \lambda I) = 4 - \lambda = 0 \\ \Rightarrow \lambda = 4$$

and

$$\det(A^T A - \lambda I) = \dots = \lambda^3(\lambda - 4) = 0 \\ \Rightarrow \lambda \in \{0, 4\}$$

For  $\lambda = 4$  we get one eigenvector  $v_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ . For  $\lambda = 0$  we get three eigenvectors

with only one constraint, that the sum of their entries is zero. We choose them to be orthogonal to each other and normalize them, so that matrix  $V$  is indeed unitary.

$$v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad v_4 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$AA^T$  has one eigenvalue  $\lambda = 4$ , therefore  $\sigma = 2$  and  $D = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}$  since  $A$  has rank 1.

We now can write the SVD of  $A$ :

$$A = UDV^* = [1] \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix} c \begin{bmatrix} c & c & c & c \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ c & c & -c & -c \end{bmatrix}$$

where  $c = \frac{1}{\sqrt{2}}$ .

The pseudoinverse of  $A$  is then

$$A^+ = VDU^* = c \begin{bmatrix} c & 1 & 0 & c \\ c & -1 & 0 & c \\ c & 0 & 1 & -c \\ c & 0 & -1 & -c \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} [1] = \frac{c^2}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

For  $B$  we have

$$B = UDV^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and therefore pseudoinverse

$$B^+ = VD^+U^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Finally, for  $C$  we have

$$C = UDV^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

and therefore pseudoinverse

$$C^+ = VD^+U^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$