Multiple View Geometry: Exercise Sheet 2

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Exercise: May 2nd, 2018

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. Which groups have you seen in the lecture? Write down the names and the correct inclusions! (e.g.: group A $\subset$ group B)

2. Let A be a symmetric matrix, and λ_a, λ_b eigenvalues with eigenvectors v_a and v_b. Prove: if v_a and v_b are not orthogonal, it follows: $\lambda_a = \lambda_b$.

   Hint: What can you say about $\langle Av_a, v_b \rangle$?

3. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with the orthonormal basis of eigenvectors $v_1, \ldots, v_n$ and eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Find all vectors x, that minimize the following term:

   $$\min_{||x||=1} x^\top A x$$

   How many solutions exist? How can the term be maximized?

   Hint: Use the expression $x = \sum_{i=1}^{n} \alpha_i v_i$ with coefficients $\alpha_i \in \mathbb{R}$ and compute appropriate coefficients!

4. Let $A \in \mathbb{R}^{m\times n}$. Prove that $\text{kernel}(A) = \text{kernel}(A^\top A)$.

   Hint: Consider a) $x \in \text{kernel}(A) \Rightarrow x \in \text{kernel}(A^\top A)$
   and b) $x \in \text{kernel}(A^\top A) \Rightarrow x \in \text{kernel}(A)$.

5. Singular Value Decomposition (SVD)

   Let $A = USV^\top$ be the SVD of A. What do you know about the properties of A, U, S, V?

   (a) Write down possible dimensions for A, U, S and V.

   (b) What are the similarities and differences between the SVD and the eigenvalue decomposition?

   (c) What do you know about the relationship between U, S, V and the eigenvalues and eigenvectors of $A^\top A$ and $AA^\top$?

   (d) What is the interpretation of the entries in S and what do the entries of S tell us about A?

Part II: Practical Exercises

The Moore-Penrose pseudo-inverse

To solve the linear system $Ax = b$ for an arbitrary (non-quadratic) matrix $A \in \mathbb{R}^{m \times n}$ of rank $r \leq \min(m, n)$, one can define a (generalized) inverse, also called the *Moore-Penrose pseudo-inverse* (compare Chapter 1, last slide).
In this exercise we want to solve the linear system $Dx = b$ with $D = [d_1, d_2, d_3, d_4]$ and $b = 1$.

1. Create some data
   (a) Let the initial linear system be the following: $4d_1 - 3d_2 + 2d_3 - d_4 = 1$.
   (b) Generate a data set consisting of 20 samples for each of the 4 variables d_1, d_2, d_3, d_4.
   (Hint: Use `rand` to define d_1, d_2, d_3 and set $d_4 = 4d_1 - 3d_2 + 2d_3 - 1$.)
   (c) Introduce small errors into the data.
   (Hint: Use `eps*rand` with `eps=1.e-5`)
2. Find the coefficients x solving the system $Dx = b$
   (a) Compute the SVD for the matrix $D = [d_1, d_2, d_3, d_4]$.
   (Hint: Use `svd`)
   (b) Compute the Moore-Penrose pseudo-inverse using the result from (a).
   (c) Compute the coefficients x.
3. Read the last slide of Chapter 1 again. Discuss with your neighbor why the following statement holds:
   $x_{min} = A^+b$ *is among all minimizers of* $|Ax - b|^2$ *the one with the smallest norm* $|x|$.