



Multiple View Geometry: Exercise Sheet 3

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Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. Indicate the matrices $M \in SE(3) \subset \mathbb{R}^{4 \times 4}$ representing the following transformations:
 - (a) Translation by the vector $T \in \mathbb{R}^3$.
 - (b) Rotation by the rotation matrix $R \in \mathbb{R}^{3 \times 3}$.
 - (c) Rotation by R followed by the translation T .
 - (d) Translation by T followed by the rotation R .
2. Let $M_1, M_2 \in \mathbb{R}^{3 \times 3}$. Please prove the following:

$$\begin{aligned} \mathbf{x}^\top M_1 \mathbf{x} = \mathbf{x}^\top M_2 \mathbf{x} & \quad \text{iff} \quad M_1 - M_2 \text{ is skew-symmetric} \\ \text{for all } \mathbf{x} \in \mathbb{R}^3 & \quad \quad \quad (\text{i.e. } M_1 - M_2 \in so(3)) \end{aligned}$$

Info: The group $SO(3)$ is called a **Lie group**.

The space $so(3) = \{\hat{\omega} \mid \omega \in \mathbb{R}^3\}$ of skew-symmetric matrices is called its **Lie algebra**.

3. Consider a vector $\omega \in \mathbb{R}^3$ with $\|\omega\| = 1$ and its corresponding skew-symmetric matrix $\hat{\omega}$.
 - (a) Show that $\hat{\omega}^2 = \omega\omega^\top - I$ and $\hat{\omega}^3 = -\hat{\omega}$.
 - (b) Following the result of (a), find simple rules for the calculation of $\hat{\omega}^n$ and proof your result. Distinguish between odd and even numbers n .
 - (c) Derive the Rodrigues' formula for a skew-symmetric matrix $\hat{\omega}$ corresponding to an arbitrary vector $\omega \in \mathbb{R}^3$ (i.e. $\|\omega\|$ does not have to be equal to 1):

$$e^{\hat{\omega}} = I + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|) + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos(\|\omega\|))$$

Hint: Combine your result from (b) with

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} \quad \text{and} \quad \sin(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \quad \text{and} \quad 1 - \cos(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}$$

Part II: Practical Exercises

This exercise is to be solved during the tutorial.

1. Homogeneous transformation matrices

- (a) Download the package `ex3.zip` and use `openOFF.m` to load the 3D model `model.off`.
- (b) Write a function that rotates the model around its *center* (i.e. the mean of its vertices) for given rotation angles α , β and γ around the x -, y - and z -axis. Use homogeneous coordinates and describe the overall transformation by a single matrix. The rotation matrices around the respective axes are as follows:

$$\begin{array}{ccc} \text{rotation matrix (x-axis)} & \text{rotation matrix (y-axis)} & \text{rotation matrix (z-axis)} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix} & \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix} & \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

- (c) Rotate the model first 45 degrees around the x -axis and then 120 degrees around the z -axis. Now start again by doing the same rotation around the z -axis first followed by the x -axis rotation. What do you observe?
- (d) Perform a translation in addition to the rotation. Find a suitable matrix from $SE(3)$ for this purpose and add it to your function from (c). Translate the model by the vector $(0.5 \ 0.2 \ 0.1)^\top$.

2. Twist-coordinates

- (a) Write a function which takes a vector $w \in \mathbb{R}^3$ as input and returns its corresponding element $R = e^{\hat{w}} \in SO(3) \subset \mathbb{R}^{3 \times 3}$ from the Lie group. Hence, the function will be a concatenation of the hat operator $\hat{\cdot}: \mathbb{R}^3 \rightarrow so(3)$ and the exponential mapping.
- (b) Implement another function which performs the corresponding inverse transformation and test the two functions on some examples.
- (c) Implement similar functions which calculate the transformation for twists. I.e. from $\xi \in \mathbb{R}^6$ to $e^{\hat{\xi}} \in SE(3) \subset \mathbb{R}^{4 \times 4}$ and the other way around.
- (d) How can you use Matlab's built-in functions `expm` and `logm` to achieve the same functionality (your solutions to (a)-(c) should *not* use these functions)?