Chapter 9
Variational Methods: A Short Intro

Multiple View Geometry
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Variational Methods

Variational Image Smoothing

Euler-Lagrange Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange

Overview

1. Variational Methods
2. Variational Image Smoothing
3. Euler-Lagrange Equation
4. Gradient Descent
5. Adaptive Smoothing
6. Euler and Lagrange

Variational Methods

Variational Image Smoothing

Euler-Lagrange Equation

Gradient Descent

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Variational Methods

Variational methods are a class of optimization methods. They are popular because they allow to solve many problems in a mathematically transparent manner. Instead of implementing a heuristic sequence of processing steps (as was commonly done in the 1980's), one clarifies beforehand what properties an 'optimal' solution should have.

Variational methods are particularly popular for infinite-dimensional problems and spatially continuous representations.

Particular applications are:

- Image denoising and image restoration
- Image segmentation
- Motion estimation and optical flow
- Spatially dense multiple view reconstruction
- Tracking

Variational Methods

Variational Image Smoothing

Euler-Lagrange Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange

Advantages of Variational Methods

Variational methods have many advantages over heuristic multi-step approaches (such as the Canny edge detector):

- A mathematical analysis of the considered cost function allows to make statements on the existence and uniqueness of solutions.
- Approaches with multiple processing steps are difficult to modify. All steps rely on the input from a previous step. Exchanging one module by another typically requires to re-engineer the entire processing pipeline.
- Variational methods make all modeling assumptions transparent, there are no hidden assumptions.
- Variational methods typically have fewer tuning parameters. In addition, the effect of respective parameters is clear.
- Variational methods are easily fused – one simply adds respective energies / cost functions.

Variational Methods

Variational Image Smoothing

Euler-Lagrange Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange

Example: Variational Image Smoothing

Let $f : \Omega \to \mathbb{R}$ be a grayvalue input image on the domain $\Omega \subset \mathbb{R}^2$. We assume that the observed image arises by some 'true' image corrupted by additive noise. We are interested in a denoised version u of the input image f.

The approximation u should fulfill two properties:

- It should be as similar as possible to f.
- It should be spatially smooth (i.e. 'noise-free').

Both of these criteria can be entered in a cost function of the form

$$E(u) = E_{data}(u, f) + E_{smoothness}(u)$$

The first term measures the similarity of f and u. The second one measures the smoothness of the (hypothetical) function u.

Most variational approaches have the above form. They merely differ in the specific form of the data (similarity) term and the regularity (or smoothness) term.

Variational Methods
Variational Image Smoothing
Euler-Lagrange Equation
Gradient Descent
Adaptive Smoothing
Euler and Lagrange

Variational Methods

Variational Image Smoothing

Euler-Lagrange Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange

Example: Variational Image Smoothing

For denoising a grayvalue image $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, specific examples of data and smoothness term are:

$$E_{data}(u, f) = \int_\Omega \big(u(x) - f(x)\big)^2 \, dx,$$

and

$$E_{smoothness}(u) = \int_\Omega |\nabla u(x)|^2 \, dx,$$

where $\nabla = (\partial/\partial x, \partial/\partial y)^\top$ denotes the spatial gradient.

Minimizing the weighted sum of data and smoothness term

$$E(u) = \int \big(u(x) - f(x)\big)^2 \, dx \; + \; \lambda \int |\nabla u(x)|^2 \, dx, \quad \lambda > 0,$$

leads to a smooth approximation $u : \Omega \to \mathbb{R}$ of the input image.

Such energies which assign a real value to a function are called a functionals. How does one minimize functionals where the argument is a function $u(x)$ (rather than a finite number of parameters)?

Functional Minimization & Euler-Lagrange Equation

- As a necessary condition for minimizers of a functional the associated Euler-Lagrange equation must hold. For a functional of the form

$$E(u) = \int \mathcal{L}(u, u')\, dx,$$

it is given by

$$\boxed{\frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u'} = 0}$$

- The central idea of variational methods is therefore to determine solutions of the Euler-Lagrange equation of a given functional. For general non-convex functionals this is a difficult problem.
- Another solution is to start with an (appropriate) function $u_0(x)$ and to modify it step by step such that in each iteration the value of the functional is decreased. Such methods are called descent methods.

Variational Methods

Variational Image Smoothing

Euler-Lagrange Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange

Gradient Descent

One specific descent method is called gradient descent or steepest descent. The key idea is to start from an initialization $u(x, t = 0)$ and iteratively march in direction of the negative energy gradient.

For the class of functionals considered above, the gradient descent is given by the following partial differential equation:

$$\begin{cases} u(x,0) = u_0(x) \\ \dfrac{\partial u(x,t)}{\partial t} = -\dfrac{dE}{du} = -\dfrac{\partial \mathcal{L}}{\partial u} + \dfrac{d}{dx}\dfrac{\partial \mathcal{L}}{\partial u'}. \end{cases}$$

Specifically for $\mathcal{L}(u, u') = \frac{1}{2}\big(u(x) - f(x)\big)^2 + \frac{\lambda}{2}|u'(x)|^2$ this means:

$$\frac{\partial u}{\partial t} = (f - u) + \lambda u''.$$

If the gradient descent evolution converges: $\partial u / \partial t = -\frac{dE}{du} = 0$, then we have found a solution for the Euler-Lagrange equation.

Variational Methods
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Euler-Lagrange Equation
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Image Smoothing by Gradient Descent

$$E(u) = \int (f - u)^2 dx + \lambda \int |\nabla u|^2 \, dx \to \text{min.}$$

$$E(u) = \int |\nabla u|^2 \, dx \to \text{min.}$$

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Variational Methods

Variational Image Smoothing

Euler-Lagrange Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange

Discontinuity-preserving Smoothing

$$E(u) = \int |\nabla u|^2 \, dx \rightarrow \text{min.}$$

$$E(u) = \int |\nabla u| \, dx \rightarrow \text{min.}$$

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Variational Methods

Variational Image Smoothing

Euler-Lagrange Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange

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Variational Methods

Variational Image Smoothing

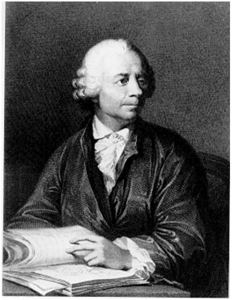
Euler-Lagrange Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange

Leonhard Euler



Leonhard Euler (1707 – 1783)

- Published 886 papers and books, most of these in the last 20 years of his life. He is generally considered the most influential mathematician of the 18th century.
- Contributions: Euler number, Euler angle, Euler formula, Euler theorem, Euler equations (for liquids), Euler-Lagrange equations,...
- 13 children

Variational Methods

Variational Image Smoothing

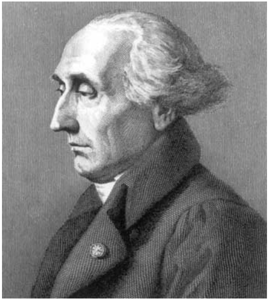
Euler-Lagrange Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange

Joseph-Louis Lagrange



Joseph-Louis Lagrange (1736 – 1813)

- born Giuseppe Lodovico Lagrangia (in Turin). Autodidact.
- At the age of 19: Chair for mathematics in Turin.
- Later worked in Berlin (1766-1787) and Paris (1787-1813).
- 1788: *La Méchanique Analytique*.
- 1800: *Leçons sur le calcul des fonctions*.

Variational Methods

Variational Image Smoothing

Euler-Lagrange Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange