Multiple View Geometry: Exercise Sheet 1

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Part I: Theory

1. Show for each of the following sets (1) whether they are linearly independent, (2) whether they span $\mathbb{R}^3$ and (3) whether they form a basis of $\mathbb{R}^3$:

   (a) $B_1 = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

   The set B_1 (1) is linearly independent, (2) spans $\mathbb{R}^3$, (3) forms a basis of $\mathbb{R}^3$.
   This can be shown by building a matrix and calculating the determinant:

   $$\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 1 \neq 0\,.$$

   As the determinant is not zero, we know that the vectors are linearly independent. Three linearly independent vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$. A set is a basis of $\mathbb{R}^3$ if it is linearly independent and spans $\mathbb{R}^3$, so B_1 forms a basis.

   (b) $B_2 = \left\{ \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$

   The set B_2 (1) is linearly independent, (2) does not span $\mathbb{R}^3$, (3) does not form a basis of $\mathbb{R}^3$.
   Since the two vectors are not parallel, linear independence is given. To span $\mathbb{R}^3$, there are at least three vectors needed. Hence, the set cannot be a basis either.

   (c) $B_3 = \left\{ \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$

   The set B_3 (1) is not linearly independent, (2) spans $\mathbb{R}^3$, (3) does not form a basis of $\mathbb{R}^3$.
   In $\mathbb{R}^3$, there cannot be more than three independent vectors. Using e.g. the determinant, one finds that any three of the four vectors form a basis of $\mathbb{R}^3$ and thus the four together span $\mathbb{R}^3$. Since they are not linearly independent, they cannot form a basis.

2. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!

   (a) $G_1 := \left\{ A \in \mathbb{R}^{n \times n} \,|\, \det(A) \neq 0 \wedge A^\top = A \right\}$

   The set is not closed under multiplication, thus no group. To show this, one counter-example is enough: choose $n = 3$ and

   $$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{pmatrix} \in G_1\,, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in G_1 \;: \quad AB = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 0 & 12 \\ 3 & 8 & 15 \end{pmatrix} \notin G_1$$

Note (comment to a question asked during the exercise):
You can also show that if G_1 was a group, for any $A, B \in G_1$, $(AB)^\top = AB$ would have to be true, but is not. This is equivalent to saying $BA = AB$ would have to be true:

$$(AB)^\top = B^\top A^\top = BA$$

However, to show that there exist A and B in G_1 for which $AB \neq BA$ (which is an important step in the proof!), the easiest way again is to choose a concrete counter-example.

(b) $G_2 := \{A \in \mathbb{R}^{n\times n} | \det(A) = -1\}$
The set contains no neutral element, thus no group:

$$\det(\text{Id}_n) = 1 \neq -1 \quad \Rightarrow \quad \text{Id}_n \notin G_2$$

(c) $G_3 := \{A \in \mathbb{R}^{n\times n} | \det(A) > 0\}$
The set forms a group. The easiest way to show this is to show that G_3 is a subgroup of the general linear group $GL(n)$. We simply need to show that for any two elements A, B of G_3, AB^{-1} is also in G_3: [1] for $A, B \in G_3$,

$$\det(AB^{-1}) = \underbrace{\det(A)}_{>0}\underbrace{[\det(B)]^{-1}}_{>0} > 0 \quad \Rightarrow \quad AB^{-1} \in G_3$$

Thus, G_3 is a subgroup of $GL(n)$ and hence a group.

3. Prove or disprove: There exist vectors $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, which are pairwise orthogonal, i.e.

$$\forall i, j = 1, ..., 5: \quad i \neq j \implies \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$$

Assume there exist five pairwise orthogonal, non-zero vectors $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^3$. In $\mathbb{R}^3$, there are at most three linearly independent vectors. Thus, the vectors are linearly dependent, which means

$$\exists\, a_i: \quad \sum_{i=1}^{5} a_i \mathbf{v}_i = 0\,,$$

with at least one $a_i \neq 0$. Without loss of generality, assume that $a_1 = -1$, resulting in

$$\mathbf{v}_1 = a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5$$

As the vectors are assumed to be pairwise orthogonal, we can derive

$$\begin{aligned}
||\mathbf{v}_1||^2 &= \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \\
&= \langle \mathbf{v}_1, a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 \rangle = \\
&= a_2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle + a_3\langle \mathbf{v}_1, \mathbf{v}_3 \rangle + a_4\langle \mathbf{v}_1, \mathbf{v}_4 \rangle + a_5\langle \mathbf{v}_1, \mathbf{v}_5 \rangle = \\
&= 0 + 0 + 0 + 0 = 0 \\
\Rightarrow \quad \mathbf{v}_1 &= \mathbf{0}\,,
\end{aligned}$$

which contradicts the assumption of pairwise orthogonal, non-zero vectors.

[1] See e.g. `https://en.wikipedia.org/wiki/Subgroup_test` for a proof if this is not clear to you.