Multiple View Geometry: Exercise Sheet 2

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Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. Which groups have you seen in the lecture? Write down the names and the correct inclusions! (e.g.: group A $\subset$ group B)

2. Let A be a symmetric matrix, and λ_a, λ_b eigenvalues with eigenvectors v_a and v_b. Prove: if v_a and v_b are not orthogonal, it follows: $\lambda_a = \lambda_b$.

   Hint: What can you say about $\langle Av_a, v_b \rangle$?

3. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with the orthonormal basis of eigenvectors $v_1, \ldots, v_n$ and eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Find all vectors x, that minimize the following term:

   $$\min_{||x||=1} x^\top A x$$

   How many solutions exist? How can the term be maximized?

   Hint: Use the expression $x = \sum_{i=1}^{n} \alpha_i v_i$ with coefficients $\alpha_i \in \mathbb{R}$ and compute appropriate coefficients!

4. Let $A \in \mathbb{R}^{m \times n}$. Prove that $\text{kernel}(A) = \text{kernel}(A^\top A)$.

   Hint: Consider a) $x \in \text{kernel}(A) \Rightarrow x \in \text{kernel}(A^\top A)$
   and b) $x \in \text{kernel}(A^\top A) \Rightarrow x \in \text{kernel}(A)$.

5. Singular Value Decomposition (SVD)

   Let $A = USV^\top$ be the SVD of A.

   (a) Write down possible dimensions for A, U, S and V.
   (b) What are the similarities and differences between the SVD and the eigenvalue decomposition?
   (c) What do you know about the relationship between U, S, V and the eigenvalues and eigenvectors of $A^\top A$ and $AA^\top$?
   (d) What is the interpretation of the entries in S and what do the entries of S tell us about A?

Part II: Practical Exercises

The Moore-Penrose pseudo-inverse

To solve the linear system $Ax = b$ for an arbitrary (non-quadratic) matrix $A \in \mathbb{R}^{m\times n}$ of rank $r \leq \min(m, n)$, one can define a (generalized) inverse, also called the *Moore-Penrose pseudo-inverse* (compare Chapter 1, last slide).
In this exercise we want to solve the linear system $Dx = b$ with $D \in \mathbb{R}^{m\times 4}$, $b \in \mathbb{R}^m$ a vector whose components are all equal to 1, and $x^* = [4, -3, 2, -1]^T \in \mathbb{R}^4$ should be one possible solution of the linear system, i.e. for any row $[d_1, d_2, d_3, d_4]$ of D:

$$4d_1 - 3d_2 + 2d_3 - d_4 = 1$$

We recall that the set of all possible solutions is given by $S = \{x^* + v \mid v \in \text{kernel}(D)\}$.

1. Create some data
   (a) Generate such a matrix D using random values with $m = 4$ rows.
   (Hint: Use `rand` to define d_1, d_2, d_3 and set $d_4 = 4d_1 - 3d_2 + 2d_3 - 1$.)
   In general, $rank(D) = 4$, hence there is a unique solution.
   (b) Introduce small additive errors into the data.
   (Hint: Use `eps*rand` with `eps=1.e-4`)

2. Find the coefficients x solving the system $Dx = b$
   (a) Compute the SVD for the matrix D.
   (Hint: Use `svd`)
   (b) Compute the Moore-Penrose pseudo-inverse using the result from (a), and compare it to the output of the `MATLAB` function `pinv`.
   (c) Compute the coefficients x, and compare it to the true solution x^*.

3. Repeat the two previous questions, by setting m to a higher value. How is the precision impacted?

4. We assume in the following that $m = 3$, hence we have infinitely many solutions.
   (a) Solve again the linear system using questions (1) and (2).
   Thus $rank(D) = 3$ and dim(kernel(D)) = 1.
   (b) Use the function `null` to get a vector $v \in$ kernel(D).
   The set of all possible solutions is $S = \{x + \lambda v \mid \lambda \in \mathbb{R}\}$.
   (c) According to the last slide of Chapter 1, we know that the following statement holds:
   $x_{min} = A^+ b$ *is among all minimizers of* $\|Ax - b\|^2$ *the one with the smallest norm* $|x|$.

   Let $\lambda \in \mathbb{R}$, $x_\lambda = x + \lambda v$ one possible solution, and $e_\lambda = \|Dx_\lambda - b\|^2$ the associated error. Using the function `plot`, display both graphs of $\|x_\lambda\|$ and e_λ according to $\lambda \in \{-100, \ldots, 100\}$, and observe that the statement indeed holds.