



Chapter 9

Variational Methods: A Short Intro

Multiple View Geometry
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Prof. Daniel Cremers
Chair for Computer Vision and Artificial Intelligence
Departments of Informatics & Mathematics
Technical University of Munich



Variational Methods

Variational Image
Smoothing

Euler-Lagrange
Equation

Gradient Descent

Adaptive Smoothing

Euler and Lagrange

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Variational methods are a class of optimization methods. They are popular because they allow to solve many problems in a mathematically transparent manner. Instead of implementing a heuristic sequence of processing steps (as was commonly done in the 1980's), one clarifies beforehand what properties an 'optimal' solution should have.

Variational methods are particularly popular for **infinite-dimensional** problems and **spatially continuous** representations.

Particular applications are:

- Image denoising and image restoration
- Image segmentation
- Motion estimation and optical flow
- Spatially dense multiple view reconstruction
- Tracking

Advantages of Variational Methods

Variational methods have **many advantages** over heuristic multi-step approaches (such as the Canny edge detector):

- A mathematical analysis of the considered cost function allows to make statements on the **existence** and **uniqueness** of solutions.
- Approaches with multiple processing steps are difficult to modify. All steps rely on the input from a previous step. Exchanging one module by another typically requires to re-engineer the entire processing pipeline.
- Variational methods make **all modeling assumptions transparent**, there are no hidden assumptions.
- Variational methods typically have **fewer tuning parameters**. In addition, the effect of respective parameters is clear.
- Variational methods are **easily fused** – one simply adds respective energies / cost functions.



Example: Variational Image Smoothing



Let $f : \Omega \rightarrow \mathbb{R}$ be a **grayvalue input image** on the domain $\Omega \subset \mathbb{R}^2$. We assume that the observed image arises by some 'true' image corrupted by additive noise. We are interested in a **denoised version u** of the input image f .

The approximation u should fulfill two properties:

- It should be as **similar** as possible to f .
- It should be **spatially smooth** (i.e. 'noise-free').

Both of these criteria can be entered in a **cost function** of the form

$$E(u) = E_{data}(u, f) + E_{smoothness}(u)$$

The first term measures the similarity of f and u . The second one measures the smoothness of the (hypothetical) function u .

Most variational approaches have the above form. They merely differ in the specific form of the data (similarity) term and the regularity (or smoothness) term.

Example: Variational Image Smoothing

For denoising a grayvalue image $f : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, specific examples of data and smoothness term are:

$$E_{data}(u, f) = \int_{\Omega} (u(x) - f(x))^2 dx,$$

and

$$E_{smoothness}(u) = \int_{\Omega} |\nabla u(x)|^2 dx,$$

where $\nabla = (\partial/\partial x, \partial/\partial y)^\top$ denotes the **spatial gradient**.

Minimizing the weighted sum of data and smoothness term

$$E(u) = \int (u(x) - f(x))^2 dx + \lambda \int |\nabla u(x)|^2 dx, \quad \lambda > 0,$$

leads to a **smooth approximation** $u : \Omega \rightarrow \mathbb{R}$ of the input image.

Such energies which assign a real value to a function are called a **functionals**. How does one minimize functionals where the argument is a function $u(x)$ (rather than a finite number of parameters)?



Functional Minimization & Euler-Lagrange Equation



- As a **necessary condition** for minimizers of a functional the associated **Euler-Lagrange equation** must hold. For a functional of the form

$$E(u) = \int \mathcal{L}(u, u') dx,$$

it is given by

$$\boxed{\frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} = 0}$$

- The central idea of variational methods is therefore to determine **solutions of the Euler-Lagrange equation** of a given functional. **For general non-convex functionals this is a difficult problem.**
- Another solution is to start with an (appropriate) function $u_0(x)$ and to modify it step by step such that in each iteration the value of the functional is decreased. Such methods are called **descent methods**.



One specific descent method is called **gradient descent** or **steepest descent**. The key idea is to start from an initialization $u(x, t = 0)$ and iteratively march in direction of the negative energy gradient.

For the class of functionals considered above, the gradient descent is given by the following **partial differential equation**:

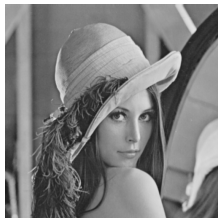
$$\begin{cases} u(x, 0) = u_0(x) \\ \frac{\partial u(x, t)}{\partial t} = -\frac{dE}{du} = -\frac{\partial \mathcal{L}}{\partial u} + \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u'} \end{cases}$$

Specifically for $\mathcal{L}(u, u') = \frac{1}{2}(u(x) - f(x))^2 + \frac{\lambda}{2}|u'(x)|^2$ this means:

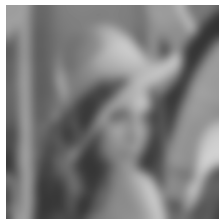
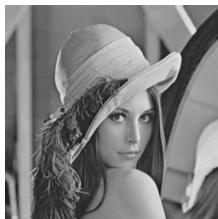
$$\frac{\partial u}{\partial t} = (f - u) + \lambda u''.$$

If the gradient descent evolution converges: $\partial u / \partial t = -\frac{dE}{du} = 0$, then we have found a solution for the Euler-Lagrange equation.

Image Smoothing by Gradient Descent



$$E(u) = \int (f - u)^2 dx + \lambda \int |\nabla u|^2 dx \rightarrow \min.$$

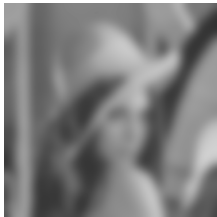
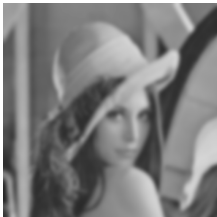


$$E(u) = \int |\nabla u|^2 dx \rightarrow \min.$$

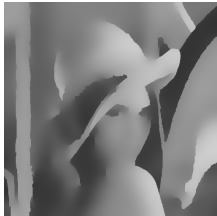
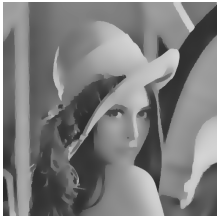
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Discontinuity-preserving Smoothing



$$E(u) = \int |\nabla u|^2 dx \rightarrow \min.$$



$$E(u) = \int |\nabla u| dx \rightarrow \min.$$

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Discontinuity-preserving Smoothing

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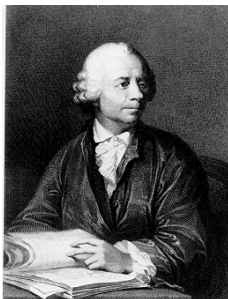
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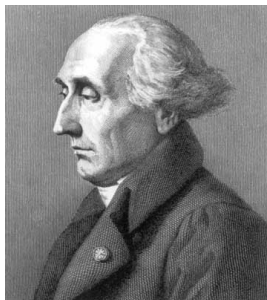
Adaptive Smoothing

Euler and Lagrange



Leonhard Euler (1707 – 1783)

- Published 886 papers and books, most of these in the last 20 years of his life. He is generally considered the most influential mathematician of the 18th century.
- Contributions: Euler number, Euler angle, Euler formula, Euler theorem, Euler equations (for liquids), Euler-Lagrange equations,...
- 13 children



Joseph-Louis Lagrange (1736 – 1813)

- born Giuseppe Lodovico Lagrangia (in Turin). Autodidact.
- At the age of 19: Chair for mathematics in Turin.
- Later worked in Berlin (1766-1787) and Paris (1787-1813).
- 1788: *La Mécanique Analytique*.
- 1800: *Leçons sur le calcul des fonctions*.