Computer Vision & Artificial Intelligence
Department of Informatics
Technical University of Munich

II : Graphical Model Representation

Tao Wu, Yuesong Shen, Zhenzhang Ye

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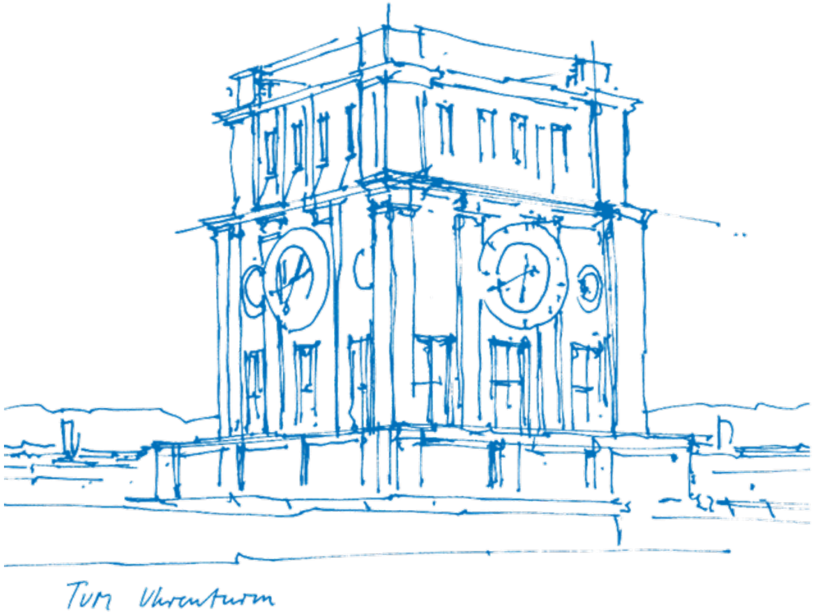
Technical University of Munich



Outline of the Chapter

- Bayesian network (directed graphical model).
- Markov random field (undirected graphical model).
- Independence assumption, representation power, parameterization, etc.

Bayesian Network

Bayesian Network (BN)

A **Bayesian network** (BN) is a *directed acyclic graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ together with:

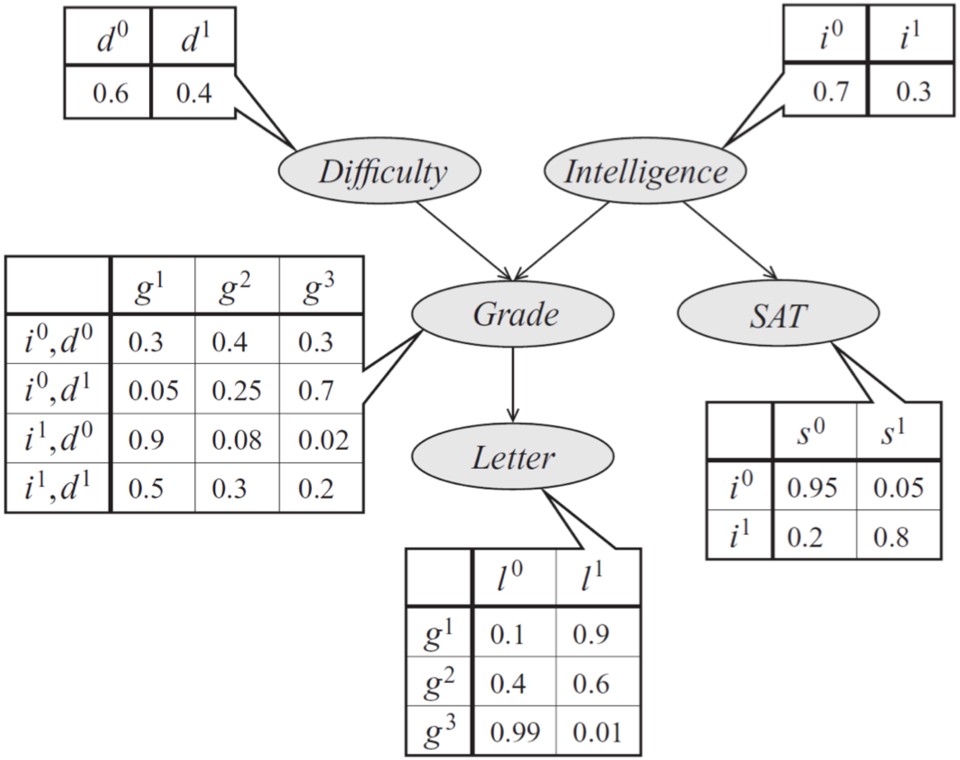
- Random variables $X = (X_i)_{i \in \mathcal{V}}$ over $\mathcal{V}$;
- A (joint probability) distribution P *factorized* as a product of conditional probability distributions (CPDs):

$$p(x) = \prod_{i \in \mathcal{V}} p\big(x_i \big| (x_j)_{j \in \mathrm{Pa}_{\mathcal{G}}(i)}\big),$$

where $\mathrm{Pa}_{\mathcal{G}}(i) = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ consists of parents of i in $\mathcal{G}$.

Example "Student"

$$P(D, I, G, S, L) = P(D)P(I)P(G|D, I)P(S|I)P(L|G).$$



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

Figure: Bayesian network represented in probability tables.

Model Complexity

Consider BN representation for RVs $(X_i)_{i=1}^n$.

- If each RV X_i takes at most d outcomes and has at most k parents, then representation of
$$p(x_i | (x_j)_{j \in \mathrm{Pa}_{\mathcal{G}}(i)})$$
requires $O(d^{k+1})$ free parameters.
- Since the joint distribution for $(X_i)_{i=1}^n$ is a product of n CPDs, the overall model complexity for BN is $O(nd^{k+1})$.
- Compared to a naive representation for the joint distribution which requires $O(d^n)$ parameters (typically $n \gg k$).

The reduction of complexity is due to the underlying independence assumptions.

Independencies in BNs

- For a distribution P for RVs (X_i), we denote by $\mathcal{I}(P)$ the set of all **independence assumptions (assertions)** that hold in P:

  $$\mathcal{I}(P) = \big\{(X_i \perp X_j \,|\, X_k)\big\}.$$

  Recall conditional independence: $X_i \perp X_j \,|\, X_k$ iff

  $$p(x_i, x_j | x_k) = p(x_i | x_k) p(x_j | x_k).$$

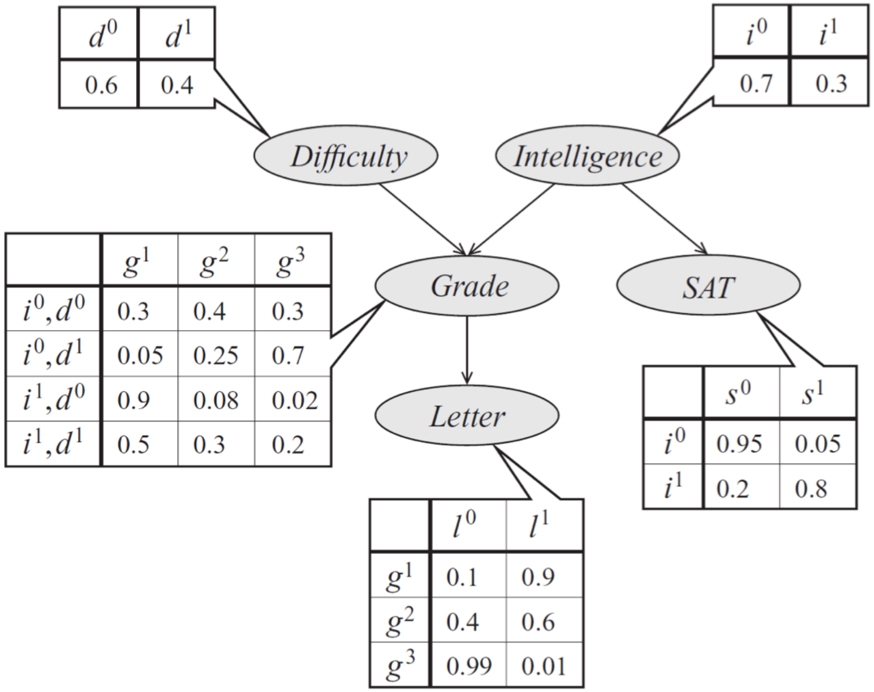
- BN $\mathcal{G}$ implies **local independencies**:

  $$\mathcal{I}_\ell(\mathcal{G}) = \Big\{\Big(X_i \perp (X_j)_{j \in \mathrm{NonDes}_\mathcal{G}(i) \setminus \{i\} \setminus \mathrm{Pa}_\mathcal{G}(i)} \,|\, (X_k)_{k \in \mathrm{Pa}_\mathcal{G}(i)}\Big)\Big\},$$

  where $\mathrm{NonDes}_\mathcal{G}(i)$ contains the non-descendants of i in $\mathcal{G}$.

Example "Student"

$$\mathcal{I}_\ell(\mathcal{G}) = \Big\{ \Big(X_i \perp (X_j)_{j\in \mathrm{NonDes}_{\mathcal{G}}(i)\setminus\{i\}\setminus \mathrm{Pa}_{\mathcal{G}}(i)} \;\big|\; (X_k)_{k\in \mathrm{Pa}_{\mathcal{G}}(i)} \Big) \Big\}.$$



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
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	l^0	l^1
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g^3	0.99	0.01

In this example we have, e.g., $(L \perp \{I, D, S\} \mid G)$, $(G \perp S \mid \{I, D\}) \in \mathcal{I}_\ell(\mathcal{G})$.

Beyond Local Independence

- Does $\mathcal{G}$ encode other independence assertions besides $\mathcal{I}_\ell(\mathcal{G})$? (Yes.)
- How to identify a specific independence assertion in $\mathcal{G}$? (D-separation.)

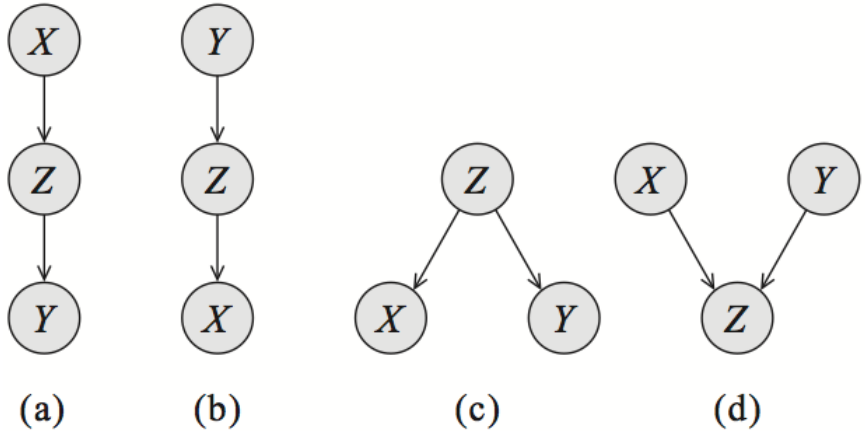
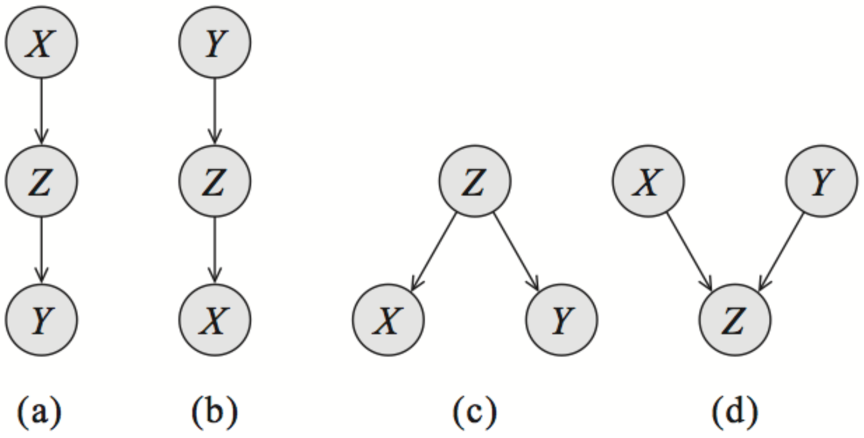


Figure: Two-edge trails from X to Y via Z. (d) is called the **v-structure**.

In the above figure, information/dependence flows from X to Y if the **trail** $X \leftrightarrow Z \leftrightarrow Y$ is **active**. This is the case if:

- In (a)–(c), Z is unobserved. (In contrast, $X \perp Y \mid Z$.)
- In (d), Z or one of its descendants is observed. (In contrast, $X \perp Y$ o.w.)

Active Trail



Let $X_1 \leftrightarrow X_2 \leftrightarrow \ldots \leftrightarrow X_n$ be a trail in a BN $\mathcal{G}$, and Z be a set of observed nodes (RVs). The trail is **active** given Z if

- Whenever there is a v-structure (case (d)) in the trail $X_{i-1} \leftrightarrow X_i \leftrightarrow X_{i+1}$, then X_i or one of its descendants are in Z.
- No other node along the trail belongs to Z.

Intuitively, information/dependence flows from X_1 to X_n (and vice versa) through the active trail $X_1 \leftrightarrow X_2 \leftrightarrow \ldots \leftrightarrow X_n$.

D-separation, Global Independence

Let X, Y, Z be three sets of nodes in a BN $\mathcal{G}$. If there is no active trail between any node in X and Y given Z, we say X and Y are **d-separated** given Z.

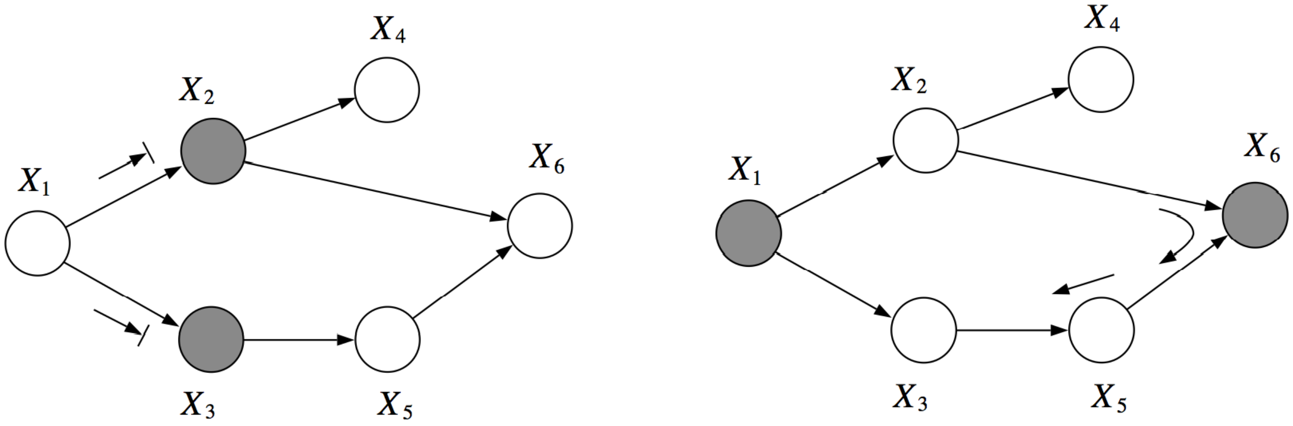


Figure: (left) X_1 and X_6 are d-sep. given $\{X_2, X_3\}$; (right) X_2 and X_3 are not d-sep. given $\{X_1, X_6\}$.

We denote by $\mathcal{I}(\mathcal{G})$ the set of **global Markov independencies**:

$$\mathcal{I}(\mathcal{G}) = \{(X \perp Y \mid Z) : X \text{ and } Y \text{ are d-separated given } Z\}.$$

Facts about D-separation

F1. (Soundness) If a distribution P factorizes according to $\mathcal{G}$, then $\mathcal{I}(\mathcal{G}) \subset \mathcal{I}(P)$. The converse is also true. In this case, we call $\mathcal{G}$ an **I-map** for P.

F2. (Sharpness) If nodes X and Y are not d-separated given Z in $\mathcal{G}$, then X and Y are dependent given Z in some distribution P that factorizes over $\mathcal{G}$.

F3. (Completeness) When a distribution P factorizes according to $\mathcal{G}$, $\mathcal{I}(\mathcal{G}) = \mathcal{I}(P)$ does not necessarily holds. Obviously, one can add superfluous edges to $\mathcal{G}$ s.t. $\mathcal{I}(\mathcal{G}) \subsetneq \mathcal{I}(P)$.

$p(b\|a)$	b_0	b_1
a_0	0.4	0.6
a_1	0.4	0.6

Figure: Here $A \perp B$. Note that $A \rightarrow B$ is an I-map for P, but $\emptyset = \mathcal{I}(\mathcal{G}) \subsetneq \mathcal{I}(P)$.

Remark: For almost all P (except for a set of measure zero in the space of CPD parameterizations) for which $\mathcal{G}$ is an I-map, we have $\mathcal{I}(\mathcal{G}) = \mathcal{I}(P)$.

I-equivalence

We can compare two BNs using their independence assertions.

- Two BNs $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be **I-equivalent** if $\mathcal{I}(\mathcal{G}_1) = \mathcal{I}(\mathcal{G}_2)$.
- The **skeleton** of a BN $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an *undirected* graph $(\mathcal{V}, \mathcal{E}')$ such that $\{X, Y\} \in \mathcal{E}'$ whenever $(X, Y) \in \mathcal{E}$.
- Fact: If two BNs have the same skeleton and the same set of v-structures, then they are I-equivalent.

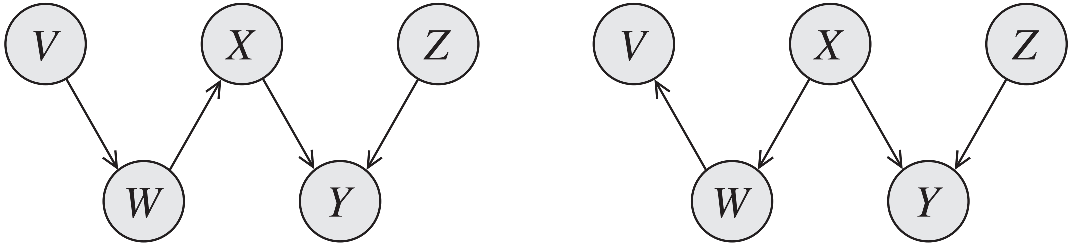


Figure: Example of two I-equivalent BNs.

Perfect Map and Counterexamples

- We say a BN $\mathcal{G}$ is a **perfect map** for a distribution P if $\mathcal{I}(\mathcal{G}) = \mathcal{I}(P)$.
- Certain independencies cannot be expressed perfectly by BN.

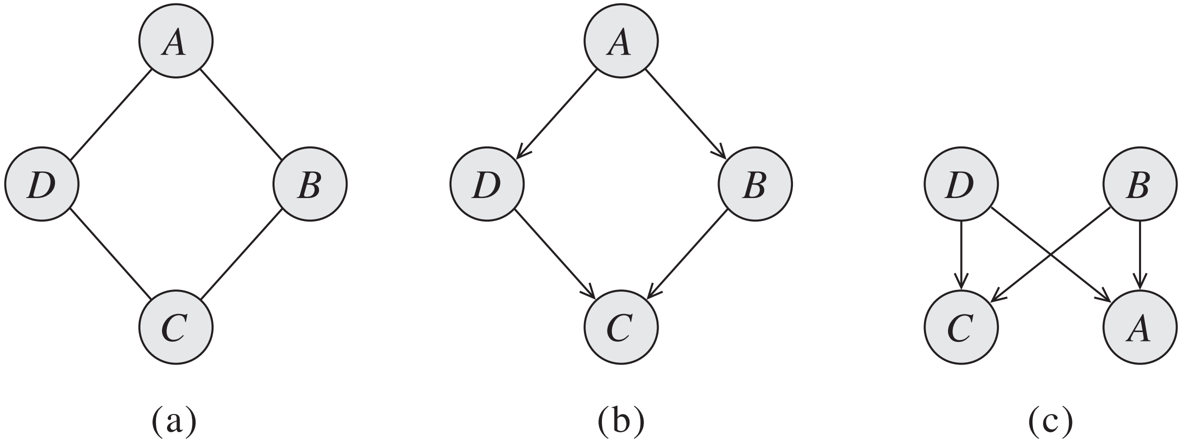


Figure: A counterexample where a perfect map does not exist.

(a) Desired independence assertions: $A \perp C \mid \{B, D\}$, $B \perp D \mid \{A, C\}$.

(b) In this BN: $(A \perp C \mid \{B, D\}) \in \mathcal{I}(\mathcal{G})$, but $(B \perp D \mid \{A, C\}) \notin \mathcal{I}(\mathcal{G})$.

(c) Again, $(A \perp C \mid \{B, D\}) \in \mathcal{I}(\mathcal{G})$, but $(B \perp D \mid \{A, C\}) \notin \mathcal{I}(\mathcal{G})$.

Topics which are not covered here ...

- Algorithm for detecting d-separation in a BN $\mathcal{G}$.
- Algorithm for finding minimal I-map $\mathcal{G}$ for a given distribution P.
- Algorithm for finding perfect map $\mathcal{G}$ (if exists) for a given distribution P.
- Further reading: Koller & Friedman, Chapter 3.

Markov Random Field

Markov Random Field (MRF)

A Markov Random Field (MRF) is an *undirected graph* $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, together with a (joint probability) distribution P for RVs $X = (X_i)_{i \in \mathcal{V}}$ s.t.

$$p(x) = \frac{1}{Z} \prod_{C \in \mathcal{C}_{\mathcal{H}}} \phi_C(x_C), \qquad (\dagger)$$

- $\mathcal{C}_{\mathcal{H}}$ is the set of **cliques** (i.e. *fully connected subgraphs*) of $\mathcal{H}$.
- Each ϕ_C is a (nonnegative) **factor** on the clique C, and $x_C = (x_i)_{i \in \mathcal{V}_C}$.
- Z is the **partition function**

  $$Z = \sum_x \prod_{C \in \mathcal{C}_{\mathcal{H}}} \phi_C(x_C),$$

  which is a normalization constant ensuring $\sum_x p(x) = 1$.

Distributions that can be factorized in form of $(\dagger)$ are called **Gibbs distributions**.

Illustration of MRF

$$p(a,b,c,d) = \frac{1}{Z}\phi_{\{A,B\}}(a,b)\phi_{\{B,C\}}(b,c)\phi_{\{C,D\}}(c,d)\phi_{\{D,A\}}(d,a),$$

$$Z = \sum_{a,b,c,d} \phi_{\{A,B\}}(a,b)\phi_{\{B,C\}}(b,c)\phi_{\{C,D\}}(c,d)\phi_{\{D,A\}}(d,a).$$

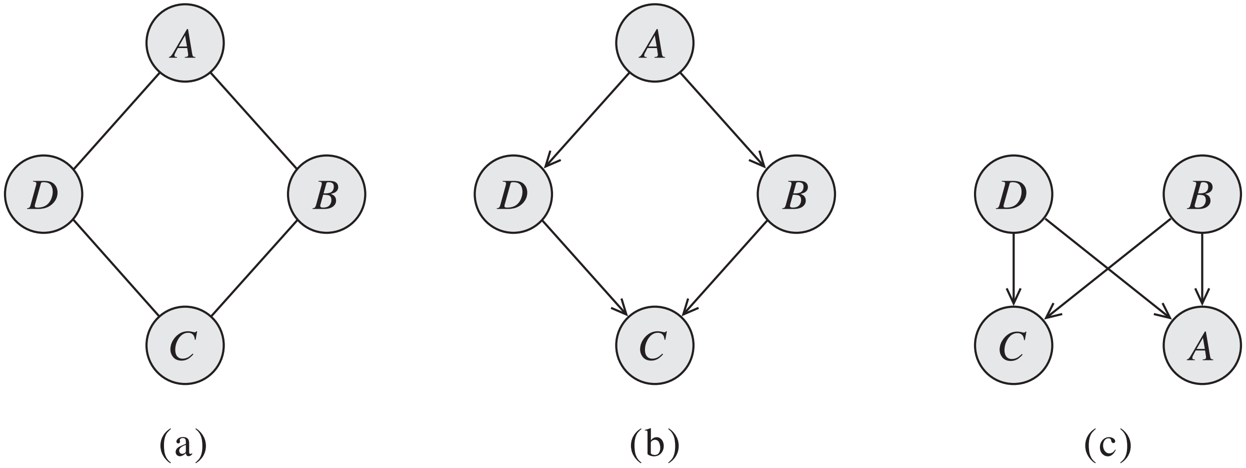


Figure: MRF in (a) cannot be perfectly represented by BN in (b) or (c).

Independencies in MRFs

Recall that global independencies in BNs are characterized by "active trail" and "d-separation". We do the equivalent for MRFs.

- Let X_1 — ... — X_n be a path in MRF $\mathcal{H}$, and O the set of observed nodes. The path X_1 — ... — X_n is **active** given O if none of $(X_i)_{i=1}^n$ belongs to O.
- Let X, Y, O be three sets of nodes in MRF $\mathcal{H}$. If there is no active path between any node in X and Y given O, then we say X and Y are **separated** given O.
- We define the **global independencies** given by $\mathcal{H}$ as:

$$\mathcal{I}(\mathcal{H}) = \{(X \perp Y \mid O) : X \text{ and } Y \text{ are separated given } O\}.$$

Facts about Separation in MRF

F1. (Soundness) If a distribution P factorizes according to MRF $\mathcal{H}$, then $\mathcal{H}$ is an I-map for P, i.e. $\mathcal{I}(\mathcal{H}) \subset \mathcal{I}(P)$.

F2. (Hammersley-Clifford theorem) Converse to (F1), if $\mathcal{H}$ is an I-map for a *positive* distribution P, then P factorizes according to $\mathcal{H}$. (A **positive distribution** has strictly positive probability for any (non-empty) event.)

F3. (Sharpness) If nodes X and Y are not separated given O in $\mathcal{H}$, then X and Y are dependent given O in some distribution P that factorizes over $\mathcal{H}$.

F4. (Completeness) When a distribution P factorizes according to $\mathcal{H}$, $\mathcal{I}(\mathcal{H}) = \mathcal{I}(P)$ does not necessarily hold.

Markov Blanket

- Let RVs $X = (X_i)_{i \in \mathcal{V}}$ and a distribution P for X be given. the **Markov blanket** of nodes $Y \subset X$ ("$\subset$" meaning $Y = (X_i)_{i \in \mathcal{V}'}$ with $\mathcal{V}' \subset \mathcal{V}$) under P is the minimal set of nodes $U \subset X \backslash Y$ s.t.
$$\big(Y \perp X \backslash Y \backslash U \,|\, U\big) \in \mathcal{I}(P).$$
- Fact: If a distribution P factorizes according to MRF $\mathcal{H}$, then the **Markov blanket** of any node is given by its neighbors in $\mathcal{H}$.

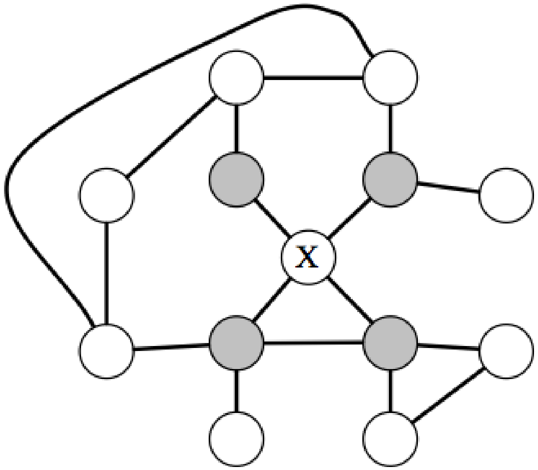


Figure: Markov blanket for node X.

Minimal I-Map (MRF case)

- One can use Markov blanket to construct "minimal" I-map.
- An I-map $\mathcal{H}$ for P is **minimal** if removing any edge from $\mathcal{H}$ renders it no longer an I-map for P. Note that a minimal I-map is not necessarily perfect, i.e. $\mathcal{I}(\mathcal{H}) = \mathcal{I}(P)$.
- Let P be a *positive* distribution for $X = (X_i)_{i \in \mathcal{V}}$. To construct a minimal I-map $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, set

$$\mathcal{E} = \left\{ \{i, j\} \in \mathcal{V} \times \mathcal{V} : X_j \text{ belongs to the Markov blanket of } X_i \text{ under } P \right\}.$$

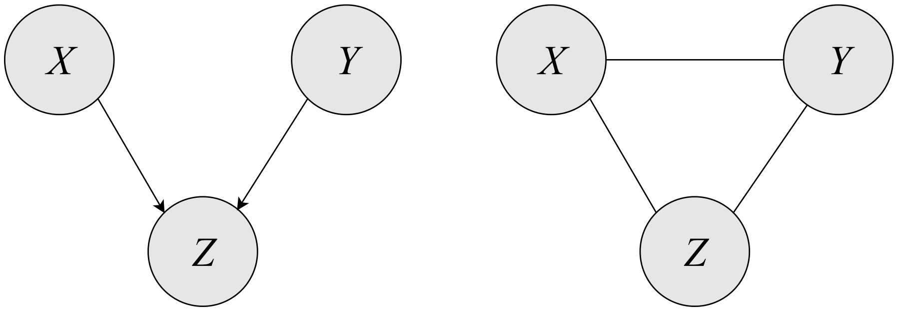


Figure: (left) P factorized according BN (the v-structure) indicates dependence of X and Y given Z observed. (right) Hence, an I-map for P by MRF must have the edge $\{X, Y\}$.

Factor Graph

In an MRF, the joint distribution is factorized into a product of factors. It is possible to make factor-node interaction explicit in a "factor graph".

- A **factor graph** is a tuple $\mathcal{G} = (\mathcal{V}, \mathcal{F}, \mathcal{E})$ consisting of a set $\mathcal{V}$ of **variable nodes**, a set $\mathcal{F} \subset 2^{\mathcal{V}}$ of **factor nodes**, and a set $\mathcal{E} \subset \mathcal{V} \times \mathcal{F}$ of **edges**.
- Each edge in $\mathcal{E}$ connects one variable node and a factor node, hence the overall factor graph $\mathcal{G}$ is **bipartite**.
- The factor graph $\mathcal{G}$ defines a family of joint distributions for $X = (X_i)_{i \in \mathcal{V}}$ factorized as

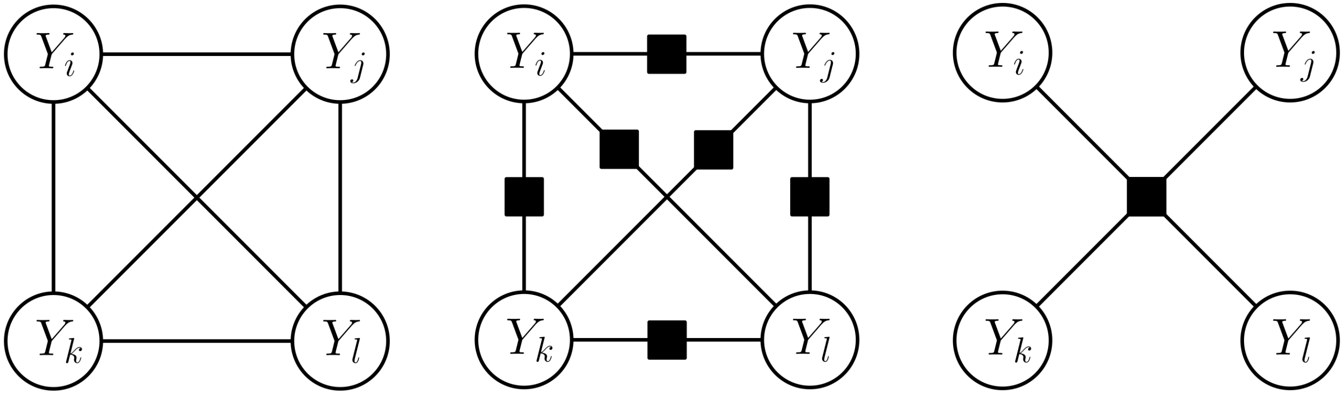
$$p(x) = \frac{1}{Z} \prod_{F \in \mathcal{F}} \phi_F(x_F),$$

$$Z = \sum_{x} \prod_{F \in \mathcal{F}} \phi_F(x_F),$$

  with each ϕ_F being a factor for $X_F = (X_i)_{i \in \mathcal{V} : (i,F) \in \mathcal{E}}$.

Illustration of Factor Graph

Figure: (left) A fully connected MRF with four nodes; (mid) Factor graph with pairwise factors; (right) Factor graph with a single joint factor.



Remark: Factor graphs in (mid) and (right) are both valid for the MRF in (left). Hence, the ambiguity in the factorization of MRF is resolved by factor graph representation.

Parameterization of MRFs

- In a factor graph, we often rewrite a factor ϕ_F using **energy function** E_F:

$$\phi_F(x_F) =: \exp(-E_F(x_F)) \quad \Rightarrow$$

$$p(x) = \exp\Big(-\sum_{F\in\mathcal{F}} E_F(x_F) - \log Z\Big),$$

$$\log Z = \log \sum_x \exp\Big(-\sum_{F\in\mathcal{F}} E_F(x_F) - \log Z\Big).$$

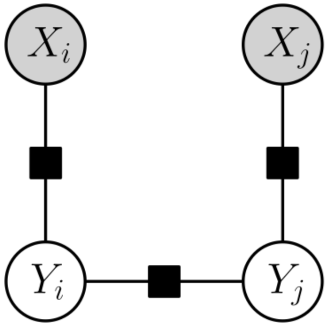
- MRF in **log-linear form** (useful for learning):

$$p(x;\theta) = \exp\Big(-\sum_{C\in\mathcal{C}_{\mathcal{H}}} \theta_C^\top \psi_C(x_C) - \log Z(\theta)\Big),$$

$$\log Z(\theta) = \log \sum_x \exp\Big(-\sum_{C\in\mathcal{C}_{\mathcal{H}}} \theta_C^\top \psi_C(x_C)\Big).$$

Each ψ_C maps x_C to a set of "features"; θ_C are weights which yield a linear function of features.

Conditional Random Field (CRF)



In some applications, a subset of nodes of an MRF are always observable. In this case, we can simplify MRF as conditional random field. A **conditional random field** (CRF) is a factor graph $\mathcal{G} = (\mathcal{V}, \mathcal{F}, \mathcal{E})$, with

- $\mathcal{V}$ consists of *observable nodes* X and *target nodes* Y.
- $\mathcal{F}$ must not contain any subset of $\mathcal{X}$.
- The conditional distribution $P(Y|X)$ is factorized as

$$p(y|x) = \frac{1}{Z(x)} \prod_{F \in \mathcal{F}} \phi_F(y_{F \cap Y}; x_{F \cap X}),$$

$$Z(x) = \sum_{y} \prod_{F \in \mathcal{F}} \phi_F(y_{F \cap Y}; x_{F \cap X}).$$

MAP Inference on CRF

CRF parameterized by energies:

$$p(y|x) = \exp\left(-\sum_{F\in\mathcal{F}} E_F(y_F; x_F) - \log Z(x)\right),$$

$$\log Z(x) = \log \sum_{y} \exp\left(-\sum_{F\in\mathcal{F}} E_F(y_F; x_F)\right).$$

MAP inference given $x, (\theta_F), (E_F)$:

$$\begin{aligned}\arg\max_{y} p(y|x) &= \arg\max_{y} \exp\left(-\sum_{F\in\mathcal{F}} E_F(y_F; x_F)\right)\\ &= \arg\min_{y} \sum_{F\in\mathcal{F}} E_F(y_F; x_F) =: E(y; x).\end{aligned}$$

Example: Image segmentation via pairwise MRF:

$$E(y; x) = \sum_{i\in\mathcal{V}} E_i(y_i; x_i) + \alpha \sum_{(i,j)\in\mathcal{E}} E_{ij}(y_i, y_j; x_i, x_j),$$

Summary

- Markov random field: definition, independence assertions.
- Factor graph: explicit representation of factors in MRF.
- Parameterization of MRF: energy function, log-linear form.
- Conditional random field.
- Further reading: Koller & Friedman, Chapter 4; Murphy, Chapter 19.