Probabilistic Graphical Models in Computer Vision

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Weekly Exercises 0

Room: 02.09.023

Wednesday, 08.05.2019, 12:15 - 14:00

The theory part was **NOT** covered yet in this Monday's lecture. You can look them after the lecture on 06.05.
This exercise sheet aims to give a warm-up on probabilistic theory and programing in python. Doing it is optional but we highly recommend you to go through all the exercises.
This sheet will **NOT** be counted into the exam bonus and you do **NOT** need to submit.
We will discuss all of them on 08.05.

Intro to Probability

Exercise 1 (0 points)**.** Consider an experiment: throwing two fair dices:

1. What is the sample space Ω?
2. Give an example of an event in this case.
3. Calculate the probability of the event you give.

Solution.

1. $\Omega = \{(i, j) : 1 \leq i, j \leq 6\}$.
2. $A = \{(i, j) : i + j = 11\}$.
3. $P(A) = \frac{|A|}{|\Omega|} = \frac{2}{36}$.

Assume we have two events A and B from a sample space Ω, we introduce some notations:

1. $A \backslash B$: A occurs and B does NOT.
2. $\bar{A}$: A does NOT occur, i.e. $\bar{A} = \Omega \backslash A$.
3. $A \cup B$: either A or B occur.
4. $A \cap B$: both A and B occur.

Exercise 2 (0 points)**.** Consider the same experiment: throwing two fair dices and find two events A and B so that following conditions are satisfied at the same time:

1. $P(A \backslash B) = P(A)$
2. $P(\bar{A}) = P(B)$
3. $P(A \cup B) = 1$
4. $P(A \cap B) = 0$.

Solution. We can choose $A = \{(i,j) : i+j \text{ is even}\}$, $B = \{(i,j) : i+j \text{ is odd}\}$.

Exercise 3 (0 Points)**.** Show following properties:

1. $P(\emptyset) = 0$.
2. $A \subset B \Rightarrow P(A) \leq P(B)$.
3. $P(A \cap B) \leq \min(P(A), P(B))$.
4. $P(A \cup B) \leq P(A) + P(B)$.
5. $P(\Omega \backslash A) = 1 - P(A)$.
6. If $(A_i)_{i \in \mathbb{N}} \in \mathscr{F}$ is set of pairwise disjoint events s.t. $\bigcup_{i \in \mathbb{N}} A_i = \Omega$, then $\sum_{i \in \mathbb{N}} P(A_i) = 1$.

Solution.
1. Since $\emptyset$ and Ω are pairwise disjoint, $P(\Omega) = P(\Omega \cup \emptyset) = P(\emptyset) + P(\Omega) = 1$. Therefore, $P(\emptyset)$ can only be 0.
2. Since A and $\bar{A} \cap B$ are disjoint, we have $P(B) = P(A \cup (\bar{A} \cap B)) = P(A) + P(\bar{A} \cap B) \geq P(A)$.
3. Since $(A \cap B) \subset A$ and $(A \cap B) \subset B$, using previous conclusion, we get $P(A \cap B) \leq \min(P(A), P(B))$.
4. Still A and $(\overline{A \cap B} \cap B)$ are pairwise disjoint and $\overline{A \cap B} \cap B \subset B$, $P(A \cup B) = P(A) + P(\overline{A \cap B} \cap B) \leq P(A) + P(B)$.
5. $P(\Omega \backslash A) = P(\bar{A}) = P(\Omega) - P(A) = 1 - P(A)$.
6. $P(\Omega) = P(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} P(A_i) = 1$

Exercise 4 (0 Points)**.** Show following properties about expectation and variance:

1. $\mathbb{E}[a] = a, \forall a$ is a constant.
2. $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
3. $\mathbb{E}[\alpha X] = \alpha \mathbb{E}[X]$, α is a constant scalar.
4. $\mathrm{Cov}[X, a] = \mathrm{Cov}[a, X] = 0$, for any constant a.
5. $\mathrm{Cov}[\alpha X, \beta Y] = \alpha\beta \mathrm{Cov}[X, Y]$. ($\alpha$, β are constant scalars)

Solution.

1. We can view this as $P(X = a) = 1$. Therefore, $\mathbb{E}[a] = a$ by definition.

2. We here show the discrete case, the continuous one can be deduced in the same way.

$$\begin{aligned}
\mathbb{E}[X+Y] &= \sum_{\omega_X \in \Omega_X} \sum_{\omega_Y \in \Omega_Y} (X(\omega_X) + Y(\omega_Y)) P(\omega_X, \omega_Y) \\
&= \sum_{\omega_X \in \Omega_X} \sum_{\omega_Y \in \Omega_Y} X(\omega_X) P(\omega_X, \omega_Y) + \sum_{\omega_X \in \Omega_X} \sum_{\omega_Y \in \Omega_Y} Y(\omega_Y) P(\omega_X, \omega_Y) \\
&= \sum_{\omega_X \in \Omega_X} X(\omega_X) \sum_{\omega_Y \in \Omega_Y} P(\omega_X | \omega_Y) P(\omega_Y) + \sum_{\omega_Y \in \Omega_Y} Y(\omega_Y) \sum_{\omega_X \in \Omega_X} P(\omega_Y | \omega_X) P(\omega_X) \\
&= \mathbb{E}[X] + \mathbb{E}[Y]
\end{aligned}$$

3. $\mathbb{E}[\alpha X] = \sum_{\omega \in \Omega} \alpha X(\omega) P(\omega) = \alpha \sum_{\omega \in \Omega} X(\omega) P(\omega) = \alpha \mathbb{E}[X]$.

4. $\mathrm{Cov}[X, a] = \mathbb{E}[aX] - \mathbb{E}[X]\mathbb{E}[a] = 0$. $\mathrm{Cov}[a, X] = \mathbb{E}[aX] - \mathbb{E}[a]\mathbb{E}[X] = 0$.

5. $\mathrm{Cov}[\alpha X, \beta Y] = \mathbb{E}[\alpha\beta XY] - \mathbb{E}[\alpha X]\mathbb{E}[\beta Y] = \alpha\beta(\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]) = \alpha\beta \mathrm{Cov}[X, Y]$.

Intro to Python

Please see *python_ basics_ template.ipynb* for details. In case you have any problems about installing notebook, please check `https://jupyter.org/install`.