Probabilistic Graphical Models in Computer Vision

Lecture: Dr. Tao Wu
Exercises: Yuesong Shen, Zhenzhang Ye
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Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 1

Room: 02.09.023

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Probability (12+6 Points)

Exercise 1 (6 Points)**.** Siegfried the ornithologist does a study on the green-speckled swallow. Since he has a huge collection of bird photographs he wants to find all images depicting a green-speckled swallow. Due to it's distinctive features it is an easy task for Eduard, Siegfried's friend and computer vision scientist, to program a green-speckled swallow detector that marks all images containing such a bird. Unfortunately the detector does not work perfectly. If the image contains a green-speckled swallow the detector marks it correctly with a chance of 99.5%. If the image does not contain a green-speckled swallow the detector marks it correctly with a chance of 99.3%. The bird is also very rare: If we randomly draw an image from the collection, there is only a chance of 0.001% that the image contains a green-speckled swallow.

1. Do a formal modeling of the experiment. What is the discrete probability space?

2. What is the probability that a green-speckled swallow is on a given image, if the detector gives a positive answer?

3. What is the probability that a green-speckled swallow is on a given image, if the detector gives a negative answer?

Solution.

1. The probability space $(\Omega, \mathcal{F}, P)$ is defined by

$$\begin{aligned}
&\Omega = \{(s,+),(s,-),(n,+),(n,-)\},\\
&\mathscr{F} = \{A \subset \Omega\},\\
&P : \mathscr{F} \to [0,1],
\end{aligned} \tag{1}$$

   where the symbols s is the image contains a green-speckled swallow, n is the image does not contain a green-speckled swallow. $+$ means detector reports a gree-speckled swallow and $-$ is inverse.

2. Therefore, we have following four events:
   $A = \{(s,+),(s,-)\}$: the image contains a green-speckled swallow

$B = \{(s,+),(n,+)\}$: the detector reports positive answer
$C = \{(s,-),(n,-)\}$: the detector reports negative answer
$D = \{(n,+),(n,-)\}$: the image does not contain a green-speckled swallow
From the text we know that

$$P(B \mid A) = 0.995,\ P(C \mid D) = 0.993,\ \text{and } P(A) = 0.00001$$

Note that $\bar{A} = D$ and $\bar{B} = C$. Therefore

$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A)P(A)}{P(B)} \\ &= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid D)P(D)} \\ &= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + (1 - P(C \mid D))(1 - P(A))} = 1.41e10^{-3}. \end{aligned} \tag{2}$$

3.

$$\begin{aligned} P(A \mid C) &= \frac{P(C \mid A)P(A)}{P(C)} \\ &= \frac{(1 - P(B \mid A))P(A)}{P(C \mid A)P(A) + P(C \mid D)P(D)} \\ &= \frac{(1 - P(B \mid A))P(A)}{(1 - P(B \mid A))P(A) + P(C \mid D)(1 - P(A))} = 5.03e10^{-8}. \end{aligned} \tag{3}$$

Exercise 2 (4 Points)**.** Assume that A, B and C are events. Assuming $P(B \mid C) \neq 0$, prove that

$$P(A \mid B \cap C) = \frac{P(B \mid A \cap C)P(A \mid C)}{P(B \mid C)}. \tag{4}$$

Solution. By applying the definition of conditional probability, we get

$$\begin{aligned} \frac{P(B \mid A \cap C)P(A \mid C)}{P(B \mid C)} &= \frac{P(A \cap B \cap C)P(A \cap C)}{P(A \cap C)P(C)} \frac{P(C)}{P(B \cap C)} \\ &= \frac{P(A \cap B \cap C)}{P(B \cap C)} \\ &= P(A \mid B \cap C) \end{aligned} \tag{5}$$

Exercise 3 (4 Points)**.** Let A, B and C be events and $P(B \cap C) > 0$, show that

$$P(A \mid C) = P(A \mid B \cap C) \Leftrightarrow P(A \cap B \mid C) = P(A \mid C)P(B \mid C). \tag{6}$$

Solution. "$\Rightarrow$", Assume that $P(A \mid C) = P(A \mid B \cap C)$. Therefore,

$$\frac{P(A \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} \Rightarrow P(A \cap B \cap C) = \frac{P(A \cap C)P(B \cap C)}{P(C)}.$$

We then have

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap C)P(B \cap C)}{P(C)P(C)} = P(A \mid C)P(B \mid C).$$

"$\Leftarrow$", Assume that $P(A \cap B \mid C) = P(A \mid C)P(B \mid C)$. Therefore,

$$\frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap C)}{P(C)} \frac{P(B \cap C)}{P(C)} \Rightarrow \frac{P(A \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)}.$$

Exercise 4 (4 Points)**.** If X and Y are independent, show following equation:

$$\mathrm{Cov}[X, Y] = 0$$

Is the converse true? Prove it or give a counter example depending on your answer.

Solution. By the definition of Cov, we only need to show that X and Y are independent $\Rightarrow E[XY] = E[X]E[Y]$.

$$\begin{aligned} E[XY] &= \sum_{\omega_X \in \Omega_X} \sum_{\omega_Y \in \Omega_Y} (X(\omega_X)Y(\omega_Y))P(\omega_X, \omega_Y) \\ &= \sum_{\omega_X \in \Omega_X} \sum_{\omega_Y \in \Omega_Y} X(\omega_X)P(\omega_X)Y(\omega_Y)P(\omega_Y) \\ &= \sum_{\omega_X \in \Omega_X} X(\omega_X)P(\omega_X) \sum_{\omega_Y \in \Omega_Y} Y(\omega_Y)P(\omega_Y) \\ &= E[X]E[Y] \end{aligned}$$

The converse is not true. Consider a random variable X which can be -1 or 1 with probablity 0.5 for each. Let Y be a random variable such that $Y = 0$ if $X = -1$ and Y is -1 or 1 with probability 0.5 for each if $X = 1$.