Probabilistic Graphical Models in Computer Vision

Lecture: Dr. Tao Wu
Exercises: Yuesong Shen, Zhenzhang Ye
Summer Semester 2019

Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 10

Room: 02.09.023
Wednesday, 24.07.2019, 12:15 - 14:00

EM algorithm and GMM (Due: 22.07) (6 Points)

NOTE: There will be a Q&A session next Wednesday. If time permits, a Q&A session will also take place after the guest lecture next Monday.

Exercise 1 (6 Points)**.** (M-step for GMM) Following the example of learning GMM with EM in the lecture slides, show that the parameters π_k, μ_k, Σ_k will be updated in the M-step as follows:

$$\pi_k^{t+1} = \frac{1}{|\mathcal{S}|} \sum_{x \in \mathcal{S}} q^t(z = k|x) \tag{1}$$

$$\mu_k^{t+1} = \frac{\sum_{x \in \mathcal{S}} \big(q^t(z = k|x) \cdot x\big)}{\sum_{x \in \mathcal{S}} q^t(z = k|x)} \tag{2}$$

$$\Sigma_k^{t+1} = \frac{\sum_{x \in \mathcal{S}} \big(q^t(z = k|x)(x - \mu_k^{t+1})(x - \mu_k^{t+1})^\top\big)}{\sum_{x \in \mathcal{S}} q^t(z = k|x)} \tag{3}$$

Solution. From the lecture we know that

$$\tilde{l}^t(\theta) = -\frac{1}{|\mathcal{S}|} \sum_{x \in \mathcal{S}} \sum_k q^t(z = k|x)\Big(\log \pi_k + \log p_G(x; \mu_k, \Sigma_k)\Big). \tag{4}$$

with (d being the dimension of x)

$$\log p_G(x; \mu_k, \Sigma_k) = \frac{d}{2} \log 2\pi - \frac{1}{2} \log |\Sigma_k| - \frac{1}{2}(x - \mu_k)^\top \Sigma_k^{-1}(x - \mu_k) \tag{5}$$

First of all, notice that the mixing parameters have the additional constraint of unit summation:

$$\sum_k \pi_k = 1. \tag{6}$$

Thus we introduce the Lagrangian La^t for $\tilde{l}^t$:

$$La^t(\theta, \lambda) = \tilde{l}^t(\theta) + \lambda(\sum_k \pi_k - 1) \tag{7}$$

$$\frac{\partial La^t}{\partial \pi_k} = 0 \Rightarrow \pi_k^{t+1} = \frac{1}{|\mathcal{S}|} \frac{\sum_{x \in \mathcal{S}} q^t(z = k|x)}{\lambda} \tag{8}$$

since $\sum_k \pi_k^{t+1} = 1$, we have that

$$\pi_k^{t+1} = \frac{1}{|\mathcal{S}|} \sum_{x \in \mathcal{S}} q^t(z = k|x), \lambda = 1. \tag{9}$$

For μ_k we have that

$$\frac{\partial La^t}{\partial \mu_k} = 0 \Rightarrow \frac{1}{|\mathcal{S}|} \sum_{x \in \mathcal{S}} \sum_k q^t(z = k|x) \frac{\partial \log p_G(x; \mu_k, \Sigma_k)}{\partial \mu_k} = 0 \tag{10}$$

Thus

$$\mu_k^{t+1} = \frac{\sum_{x \in \mathcal{S}} \left(q^t(z = k|x) \cdot x\right)}{\sum_{x \in \mathcal{S}} q^t(z = k|x)} \tag{11}$$

For Σ_k, knowing from multivariate calculus that

$$\frac{\partial \log |\Sigma_k|}{\partial \Sigma_k} = \Sigma_k^{-\top}, \frac{\partial (x - \mu_k)^\top \Sigma_k^{-1} (x - \mu_k)}{\partial \Sigma_k} = -\Sigma_k^{-\top} (x - \mu_k)(x - \mu_k)^\top \Sigma_k^{-\top} \tag{12}$$

thus

$$\frac{\partial La^t}{\partial \Sigma_k} = 0 \Rightarrow \frac{1}{|\mathcal{S}|} \sum_{x \in \mathcal{S}} \sum_k q^t(z = k|x) \frac{\partial \log p_G(x; \mu_k, \Sigma_k)}{\partial \Sigma_k} = 0 \tag{13}$$

which gives us

$$\Sigma_k^{t+1} = \frac{\sum_{x \in \mathcal{S}} \left(q^t(z = k|x)(x - \mu_k^{t+1})(x - \mu_k^{t+1})^\top\right)}{\sum_{x \in \mathcal{S}} q^t(z = k|x)} \tag{14}$$

Programming (Due:29.07) (12 Points)

In this programming exercise, you are asked to design unary energies based on coarse density estimation using GMM model trained via EM. You are also asked to use graphcut library to perform MAP inference with the energies you have designed. See the ipython file for more details.