Probabilistic Graphical Models in Computer Vision

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Weekly Exercises 2

Room: 02.09.023

Wednesday, 22.05.2019, 12:15 - 14:00

Bayesian Network (12+6 Points)

Exercise 1 (4 Points)**.** Given two boolean random variables X and Y, (*i.e.* each can only be true (1) or false (0)). Show that if $X = 0$ is independent to $Y = 0$, we have X and Y are independent.

Solution. According to the property of independence, we have $P(X = 0|Y = 0) = P(X = 0)$. Then:

$$\begin{aligned} P(X = 0) &= P(X = 0|Y = 0)P(Y = 0) + P(X = 0|Y = 1)P(Y = 1) \\ &= P(X = 0)P(Y = 0) + P(X = 0, Y = 1) \\ \Rightarrow & P(X = 0)P(Y = 1) = P(X = 0, Y = 1), \end{aligned} \tag{1}$$

which implies $X = 0$ and $Y = 1$ are independent. Analogously, we can show that $X = 1$ and $Y = 0$ are independent. Lastly, we prove that $X = 1$ and $Y = 1$ are independent.

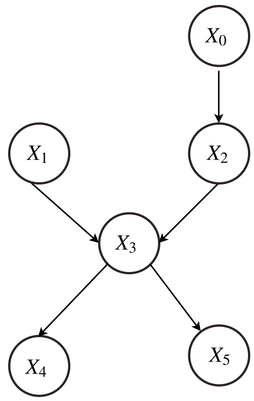
$$\begin{aligned} P(X = 1, Y = 1) &= P(X = 1|Y = 1)P(Y = 1) \\ &= (1 - P(X = 0|Y = 1))P(Y = 1) \\ &= (1 - P(X = 0))P(Y = 1) \\ &= P(X = 1)P(Y = 1) \end{aligned} \tag{2}$$

Exercise 2 (4 Points)**.** In the following Bayesian network:

1. Give the factorization of $p(x_0, x_1, x_2, x_3, x_4, x_5)$.
2. Assume the observation is $\{X_4\}$, give reachable nodes of $\{X_0\}$ via active trail.
3. Assume the observation is $\{X_0\}$, give reachable nodes of $\{X_2\}$ via active trail.

Solution.

1. $p(x_0, x_1, x_2, x_3, x_4, x_5) = p(x_0)p(x_1)p(x_2|x_0)p(x_3|x_2, x_1)p(x_4|x_3)p(x_5|x_3)$
2. Since we have a v-structure on X_3 and X_4 is observed, the reachable nodes are $\{X_0, X_1, X_2, X_3, X_5\}$.
3. Since X_3 or its descendants is not obeserved, the reachable nodes are $\{X_2, X_3, X_4, X_5\}$.



Exercise 3 (4 Points)**.** Given three boolean random variables X, Y and Z, is it possible to find a perfect map for following distribution:

$$p(x, y, z) = \begin{cases} \frac{1}{12} & x \oplus y \oplus z = \text{false} \\ \frac{1}{6} & x \oplus y \oplus z = \text{true} \end{cases} \tag{3}$$

where $\oplus$ is the XOR function. Explain why or draw the corresponding perct map.

Solution. We cannot find a perfect map for this distribution. First of all, we can write down the probability distribution for X, Y, Z. Therefore, $p(x, y) = p(x)p(y)$

X	Y	Z	$p(x, y, z)$
F	F	F	$\frac{1}{12}$
F	F	T	$\frac{1}{6}$
F	T	F	$\frac{1}{6}$
F	T	T	$\frac{1}{12}$
T	F	F	$\frac{1}{6}$
T	F	T	$\frac{1}{12}$
T	T	F	$\frac{1}{12}$
T	T	T	$\frac{1}{6}$

which implies X and Y are independent and Z is not independent of X given Y or of Y given X. Hence, we create the network $X \to Z \leftarrow Y$. However, we can also show that X and Z are independent which is not included in this graph.

Exercise 4 (6 Points)**.** For a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, assume that the max indegree is 2 (*i.e.* any node has maximum 2 parents). Given an observation set Z and a random variable Y, figure out an algorithm to find the reachable nodes of Y via active trail in this graph.
Hint: First of all, assume there is no v-structure in the graph, what should we do? Then include the v-structure, what kind of preprocessing do we have to do?

Solution. If there is no v-structure, we can perform a graph traverse algorithm directly (assume it is BFS(breadth-first search)). Once we occur an observation node, we stop that path.

For v-structure, a path is not blocked if it is the “middle“ node. This requires us to record all the ancestor nodes of Z.
So we use one algorithm which has linear time in the size of the graph. It has two phases: 1) traverse the graph, mark all nodes that are in Z or that have descendants in Z. 2) Once we get a blocked node, we stop on that path. The algorithms is shown as bellow:

Input: Graph $\mathcal{G}$, Start node Y, Observation Z
Output: R, Nodes in the active trail

```
// Phase 1: Insert all ancestors of Z into A
L ← Z ;                                          // Nodes to be visited
A ← ∅ ;                                          // Ancestors of Z
while L ≠ ∅ do
    Selcet some X from L, L ← L − {X}
    if X ∉ A then
        L ← L ∪ X_p ;                            // X's parents need to be visited
    end
    A ← A ∪ {X} ;                                // X is ancestor of observation
end
// Pahse 2: traverse active trails starting from Y
L ← {(Y,↑)} ;                // (Node, direction) to be visited, ↑ means parent
V ← ∅ ;                      // (Node, direction) marked as visited
R ← ∅ ;                      // Nodes reachable via active trail
while L ≠ ∅ do
    Select some (X,d) from L, L ← L − {(X,d)}
    if (X,d) ∉ V then
        if X ∉ Z then
            R ← R ∪ {X} ;                        // X is reachable
        end
        V ← V ∪ {(X,d)} ;                        // mark (X,d) as visited
        if d =↑ and X ∉ Z then ;                 // up on X if X is not in Z

            foreach P ∈ X_p do
                L ← L ∪ {(P,↑)} ;                // X's parents visited from bottom
            end
            foreach P ∈ X_c do
                L ← L ∪ {(P,↓)} ;                // X's children visited from top
            end
        end
        else if d =↓ then ;                      // go down on X

            if X ∉ Z then
                foreach P ∈ X_c do
                    L ← L ∪ {(P,↓)} ;            // X's children visited from top
                end
            end
            if X ∈ A then ;                      // active trail on v-structure

                foreach P ∈ X_p do
                    L ← L ∪ {(P,↑)} ;            // X's parents visited from bottom
                end
            end
        end
    end
end
return R
```

Programming (Due:27.05) (12Points)

Exercise 5. In this programmming exericse, you are asked to implement an algorithm to find reachable nodes via active trail in a directed graph. See the ipython file for more details.