Probabilistic Graphical Models in Computer Vision

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Weekly Exercises 3

Room: 02.09.023
Wednesday, 29.05.2019, 12:15 - 14:00

Markov Random Field (12+6 Points)

Exercise 1 (6 Points)**.** Given two Bayesian network graphs G_1 and G_2, explain following statements for G_1 and G_2 are I-equivalent.

1. "G_1 and G_2 has the same skeleton" is a necessary but not sufficient condition.
2. "G_1 and G_2 has the same skeleton and same v-structure" is a sufficient but not necessary condition.

Solution.

1. This is a necessary but not sufficient condition.
   - Necessity: assume G_1 and G_2 are I-equivalent but don't have the same skeleton. Thus, the extra trail of one graph will introduce extra independence that contradicts the fact that G_1 and G_2 are I-equivalent.
   - Non-sufficiency: Consider $A \to B \leftarrow C$ and $A \leftarrow B \to C$.
2. This is a sufficient but not necessary condition.
   - Sufficiency: assume that G_1 and G_2 has the same skeleton and same v-structure. For $X \perp Y | Z \in I(G_1)$, we show $X \perp Y | Z \in I(G_2)$. Since two graphs have the same skeleton, they must have same trails. If we pick some trails between X and Y in G_1 that given Z is inactive, we show that this trail in G_2 is inactive too. Consider two cases: 1) the trail in G_1 is inactive because some of the nodes on the trail that are not in a v-structure are observed. Then clearly these nodes also block the trail in G_2. 2) All nodes on the trail that are not in a V-strucutre are not observed, but then for some v-structure on the trail, none of the descendents are observed. That is for every node in the v-structure such that there is a directed path in G_1 and all the nodes on the path are not observed. Consider such a directed path in G_1, then in G_2 this trail must also be directed the same and clearly all its nodes are not observed too. Therefore this path also inactivates the trail between X and Y in G_2 given Z.

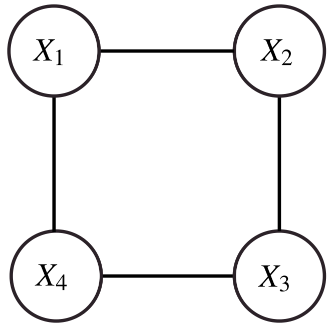

Figure 1: circle graph

- Non-necessity: for a set of nodees, construct any complete graph will lead to an empty set of conditional independence assertions, therefore these graphs are I-equivalent. But they don't have same v-structure.

Exercise 2 (6 Points)**.** In this problem, we consider a distribution which is not strictly positive and it can not be factorized by a MRF.
Assume we have four binary random variables X_i, $i \in \{1, 2, 3, 4\}$. The probability distribution assigns a probability 1/8 uniformly to each of the following set of values (X_1, X_2, X_3, X_4):

$$(0,0,0,0) \quad (1,0,0,0) \quad (1,1,0,0) \quad (1,1,1,0)$$
$$(0,0,0,1) \quad (0,0,1,1) \quad (0,1,1,1) \quad (1,1,1,1)$$

and assigns zero to all other configurations of (X_1, X_2, X_3, X_4).

1. We first show that $p(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4) = p(X_1 = x_4, X_2 = x_3, X_3 = x_2, X_4 = x_1)$.

2. Show that the distribution satisfies the global independencies with respect to the circle graph in Figure 1.
Hint: show that whatever pair of values for $\{X_2, X_4\}$, the value of X_1 or X_3 is known. Then use the conclusion from previous sub problem.

3. Show that we cannot find any factorization for $p(x_1, x_2, x_3, x_4)$.
Hint: Try to find a contradiction by examing all $\phi_{ij}(x_i, x_j)$ with (i, j) are the edges in circle graph in Figure 1.

Solution. 1. From the distribution, we can have following equations:

$$\begin{aligned}
p(0,0,0,1) &= p(1,0,0,0) = 1/8 \\
p(0,0,1,1) &= p(1,1,0,0) = 1/8 \\
p(0,1,0,1) &= p(1,0,1,0) = 0 \\
p(0,1,1,1) &= p(1,1,1,0) = 1/8 \\
p(0,0,1,0) &= p(0,1,0,0) = 0 \\
p(0,1,0,0) &= p(0,0,1,0) = 0 \\
p(1,0,1,1) &= p(1,1,0,1) = 0 \\
p(1,1,0,1) &= p(1,0,1,1) = 0
\end{aligned}$$

These equations shows that $p(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4) = p(X_1 = x_4, X_2 = x_3, X_3 = x_2, X_4 = x_1)$.

2. If $\{X_2 = 0, X_4 = 0\}$, we have $X_3 = 0$. If $\{X_2 = 0, X_4 = 1\}$, $X_1 = 0$. If $\{X_2 = 1, X_4 = 0\}$, $X_1 = 1$ and if $\{X_2 = 1, X_4 = 1\}$, $X_3 = 1$. Therefore, X_1 and X_3 are independent given $\{X_2, X_4\}$. Using the conclusion from previous problem, we can conclude that X_2 and X_4 are independent given $\{X_1, X_3\}$. This circle graph is a I-map of the distribution.

3. However, we cannot find any factorization for $p(x_1, x_2, x_3, x_4)$. Because we have:

$$\phi_{12}(0,0)\phi_{23}(0,1)\phi_{34}(1,0)\phi_{41}(0,0) = 0$$
$$\phi_{12}(0,0)\phi_{23}(0,0)\phi_{34}(0,0)\phi_{41}(0,0) = 1/8$$
$$\phi_{12}(0,0)\phi_{23}(0,1)\phi_{34}(1,1)\phi_{41}(1,0) = 1/8$$

which states that $\phi_{34}(1,0)$ must be 0. However, we know that $\phi_{12}(1,1)\phi_{23}(1,1)\phi_{34}(1,0)\phi_{41}(0,1) = 1/8$, which is a contradiction.

Exercise 3 (6 Points). Let G be a factor graph for a Markov random field consisting of N^2 binary variables, representing the pixels of an $N \times N$ image. For each piexel there is unary potential, and there are pairwise potentials according to the 8-connected neighborhood.

1. Draw the factor graph for $N = 3$.

2. What is the total number of factors, depending on N, that are included in this model.

Solution. 1. The factor graph for $N = 3$ is given in Figure **??**

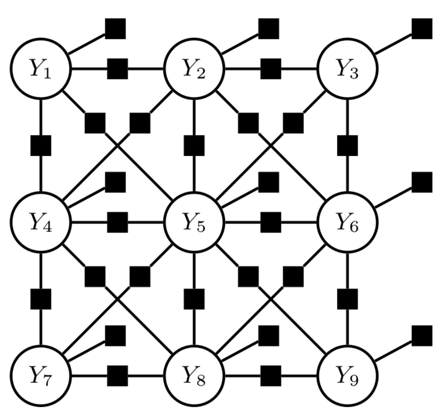

Figure 2: Factor graph

2. The total number of factors is $N^2 + 4(N-1)^2 + 2(N-1)$.