Probabilistic Graphical Models in Computer Vision

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# Weekly Exercises 6

Room: 02.09.023

Wednesday, 26.06.2019, 12:15 - 14:00

## Sampling and MCMC (Due: 24.06) (8+4 Points)

**Exercise 1** (4 Points)**.** Given a random number generator following the uniform distribution $\mathcal{U}(0,1)$. How to generate samples from distribution $\mathcal{U}(a,b)$ for some $a<b$? Justify your answer.

**Solution.** The CDF of $\mathcal{U}(a,b)$ is:

$$F(x) = \begin{cases} 0, & \text{if x < a;} \\ (x-a)/(b-a), & \text{if } a \leq x \leq b; \\ 1, & \text{if x > b.} \end{cases}$$

Taking the inverse of CDF over the range $[a,b]$, we have that $F^{-1}(x) = (b-a)\cdot x + a$. Thus we have $X \sim \mathcal{U}(0,1) \Longrightarrow (b-a)\cdot X + a \sim \mathcal{U}(a,b)$.

**Exercise 2** (8 Points)**.** Derive the polar form of Box-Muller method: Let $(X_1, X_2) \sim \mathcal{N}(0, I_2)$, let $(Z_1, Z_2) \sim \frac{1}{\pi}\mathbf{1}\{z_1^2 + z_2^2 < 1\}$,

1) Let $(T,\gamma)$ be the polar transform of $(Z_1, Z_2)$, show that we have $T^2 \sim \mathcal{U}(0,1)$, $\gamma \sim \mathcal{U}(0, 2\pi)$ and that $T, \gamma$ are independent;

2) Let $(R,\theta)$ be the polar transform of $(X_1, X_2)$, show that we have $R^2 \sim \exp(1/2)$, $\theta \sim \mathcal{U}(0,2\pi)$ and that $R, \theta$ are independent;

3) Using 1) and 2), show that by setting $R^2 = -2\log T^2, \theta = \gamma$, $R, \theta$ satisfy the required form in 2) and we can derive the following sample transformation:

$$x_i = z_i \sqrt{\frac{-2\log(z_1^2+z_2^2)}{z_1^2+z_2^2}}, i \in \{1,2\} \tag{1}$$

**Solution.**

First let's prove the following result:

Let $(S,\alpha)$ be the polar transform of $(A,B)$, the determinant of the Jacobian of the change of variable $(A,B) \to (S^2, \alpha)$ is $\det(J) = \det\begin{pmatrix} \frac{\partial A}{\partial S^2} & \frac{\partial A}{\partial \alpha} \\ \frac{\partial B}{\partial S^2} & \frac{\partial B}{\partial \alpha} \end{pmatrix} = -\frac{1}{2}$.

To prove this, let $\bar{S} = S^2$, we use the fact that

$$A = S\cos(\alpha) = \bar{S}^{1/2}\cos(\alpha) \tag{2}$$
$$B = S\sin(\alpha) = \bar{S}^{1/2}\sin(\alpha) \tag{3}$$

thus

$$\frac{\partial A}{\partial \alpha} = -\bar{S}^{1/2}\sin(\alpha) \tag{4}$$
$$\frac{\partial A}{\partial \bar{S}} = -\frac{1}{2}\bar{S}^{-1/2}\cos(\alpha) \tag{5}$$
$$\frac{\partial B}{\partial \alpha} = \bar{S}^{1/2}\cos(\alpha) \tag{6}$$
$$\frac{\partial B}{\partial \bar{S}} = -\frac{1}{2}\bar{S}^{-1/2}\sin(\alpha) \tag{7}$$

and we have

$$\det(J) = \det\begin{pmatrix} \frac{\partial A}{\partial S^2} & \frac{\partial A}{\partial \alpha} \\ \frac{\partial B}{\partial S^2} & \frac{\partial B}{\partial \alpha} \end{pmatrix} \tag{8}$$
$$= \det\begin{pmatrix} -\frac{1}{2}\bar{S}^{-1/2}\cos(\alpha) & -\bar{S}^{1/2}\sin(\alpha) \\ -\frac{1}{2}\bar{S}^{-1/2}\sin(\alpha) & \bar{S}^{1/2}\cos(\alpha) \end{pmatrix} \tag{9}$$
$$= -\frac{1}{2} \tag{10}$$

1) Let $\bar{T} = T^2$, we have that $\bar{T} = T^2 = Z_1^2 + Z_2^2, \sin(\gamma) = Z_2/T, \cos(\gamma) = Z_1/T$.

   Since $(Z_1, Z_2)$ samples uniformly from unit disk, we have that for $\bar{t} \in [0,1], \alpha \in [0, 2\pi], p(\bar{T} = \bar{t}, \gamma = \alpha) = |\det(J)| \cdot \frac{1}{\pi} \cdot 1 = \frac{1}{2\pi} \cdot 1$. Thus $\bar{T}, \gamma$ are independent and $\bar{T} = T^2 \sim \mathcal{U}(0,1)$, $\gamma \sim \mathcal{U}(0, 2\pi)$;

2) Similarly, let $\bar{R} = R^2$ we have that $\bar{R} = X_1^2 + X_2^2, \sin(\theta) = X_2/R, \cos(\theta) = X_1/R$.

   Since $p(x,y) = \frac{1}{2\pi}\exp(-(x_1^2 + x_2^2)/2) = \frac{1}{2\pi} \cdot |\det(J)|\exp(-\bar{r}/2) = p(\theta = \beta)p(\bar{R} = \bar{r})$, we have that $\bar{R}, \theta$ are independent and $\bar{R} = R^2 \sim \exp(1/2)$, $\theta \sim \mathcal{U}(0, 2\pi)$;

3) Let $(Z_1, Z_2) \sim \frac{1}{\pi}\mathbf{1}\{z_1^2 + z_2^2 < 1\}$, we know from 1) that its polar form $(T, \gamma)$ have independent coordinates and $T^2 \sim \mathcal{U}(0,1)$, $\gamma \sim \mathcal{U}(0, 2\pi)$.

   Thus $\theta = \gamma \sim \mathcal{U}(0, 2\pi)$ and the inverse CDF of exponential distribution gives us $R^2 = -2\log(T^2) \sim \exp(1/2)$ since $1 - T^2 \sim \mathcal{U}(0,1)$. $R, \theta$ are independent since $R$ only depend on $T$, $\theta$ only on $\gamma$ and that $T, \gamma$ are independent. From 2) we then have that the Cartesian coordinates

$$\begin{cases} X_1 = R\cos(\theta) = \sqrt{-2\log(T^2)} \cdot \frac{Z_1}{T} \\ X_2 = R\sin(\theta) = \sqrt{-2\log(T^2)} \cdot \frac{Z_2}{T} \end{cases} \tag{11}$$

   are independent standard Gaussian distributions. Since $T^2 = Z_1^2 + Z_2^2$, we obtain the desired transformation.