Multiple View Geometry: Exercise Sheet 10

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Part I: Theory

1. **Variational Calculus and Euler-Lagrange**

   Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth scalar function and $E(u)$ an energy functional given by

   $$E(u) = \int_\Omega \mathcal{L}\left(u(x), \nabla u(x)\right) \mathrm{d}x \,.$$

   The Gâteaux derivative of E at u in direction $h\colon \mathbb{R}^n \to \mathbb{R}$ is given by

   $$\left.\frac{\mathrm{d}E(u)}{\mathrm{d}u}\right|_h := \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[E(u+\epsilon h) - E(u)\right] = \int_\Omega \frac{\mathrm{d}E(u)}{\mathrm{d}u}(x) \cdot h(x) dx \,.$$

   (a) Under the assumption that h vanishes at the boundary of Ω, prove that

   $$\frac{\mathrm{d}E(u)}{\mathrm{d}u} = \frac{\partial \mathcal{L}(u, \nabla u)}{\partial u} - \mathrm{div}\left(\frac{\partial \mathcal{L}(u, \nabla u)}{\partial (\nabla u)}\right) \,.$$

   (b) Which condition must hold true for a minimizer u_0 of $E(u)$ ...

   - ... in general?
   - ... if $\mathcal{L}(u, \nabla u) = \mathcal{L}(u)$?
   - ... if $\mathcal{L}(u, \nabla u) = \mathcal{L}(\nabla u)$?

2. **Multiview Reconstruction as Shape Optimization**

   You saw in the lecture that 3D reconstruction from multiple views can be posed as a variational problem. Let $\rho : V \to [0,1]$ be the photoconsistency function, and $u : V \to \{0,1\}$ the indicator function of the object to be reconstructed. We want to minimize (see Chapter 10, slide 10)

   $$E(u) = \int_V \rho(x) |\nabla u(x)| \mathrm{d}x$$

   under the constraints

   $$\begin{cases} \int_{R_{ij}} u(x) \mathrm{d}R_{ij} \geq 1 & \text{if} \quad j \in S_i \,, \\ \int_{R_{ij}} u(x) \mathrm{d}R_{ij} = 0 & \text{else} \,. \end{cases}$$

   (a) Write down the Euler-Lagrange equation for the given energy $E(u)$.

   Gradient descent for energy functionals is performed in analogy to gradient descent on multivariate functions: from an estimate $u^{(k)}(x)$, the estimate in iteration $k+1$ is obtained by going in negative gradient direction:

   $$u^{(k+1)} = u^{(k)} - \tau \frac{\mathrm{d}E(u)}{\mathrm{d}u}$$

   with step size τ. This is a discretization of the differential equation from the lecture.

   (b) Write down one gradient descent iteration for $E(u)$.