Practical Course: Vision Based Navigation

Lecture 2: Camera Models and Optimization

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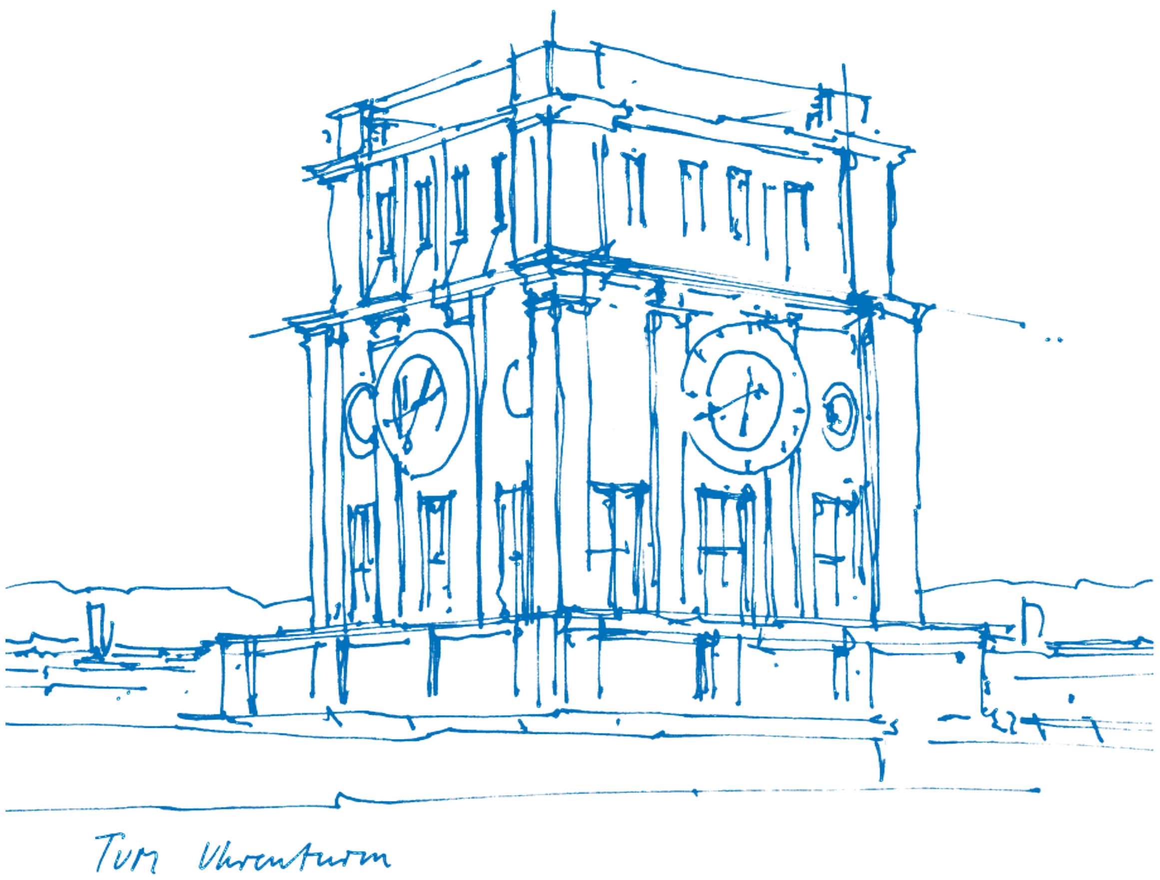
Prof. Dr. Daniel Cremers



Version: 29.04.2020

Camera Models

Image Formation

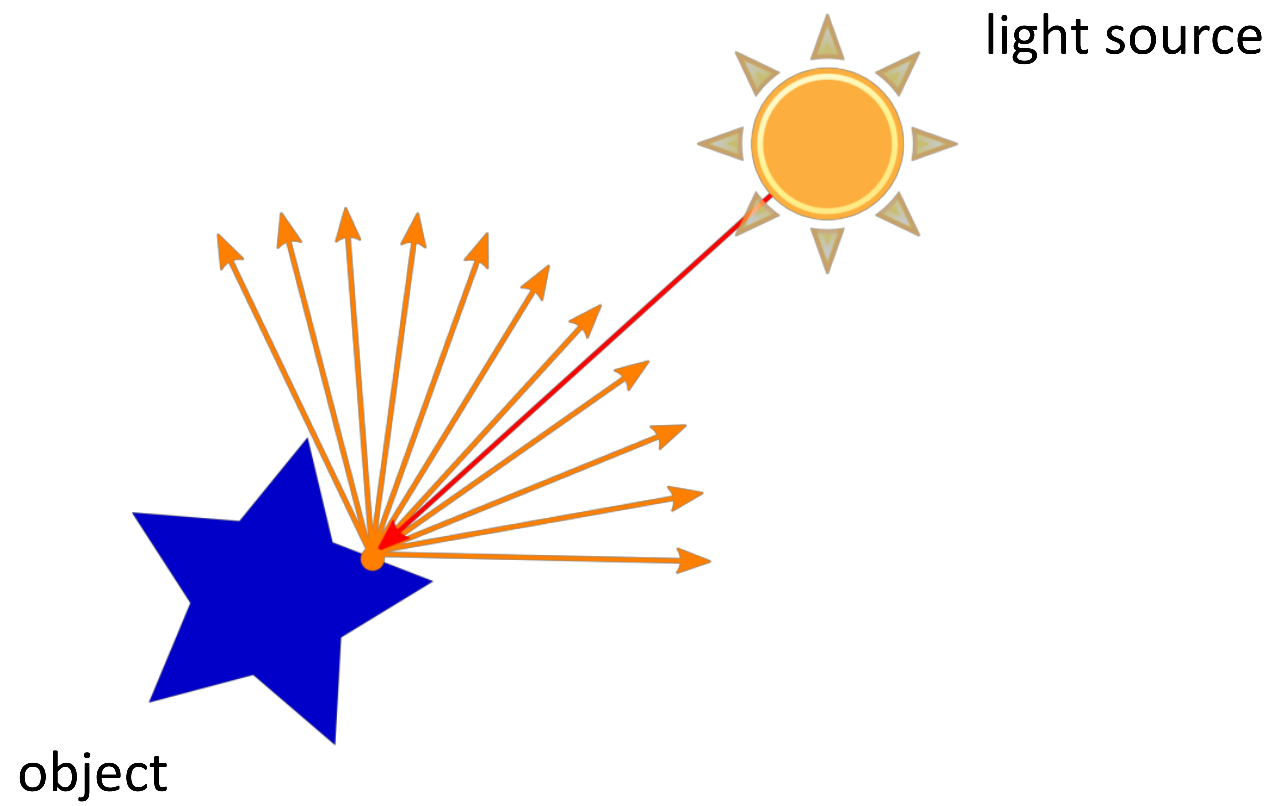


Image Formation

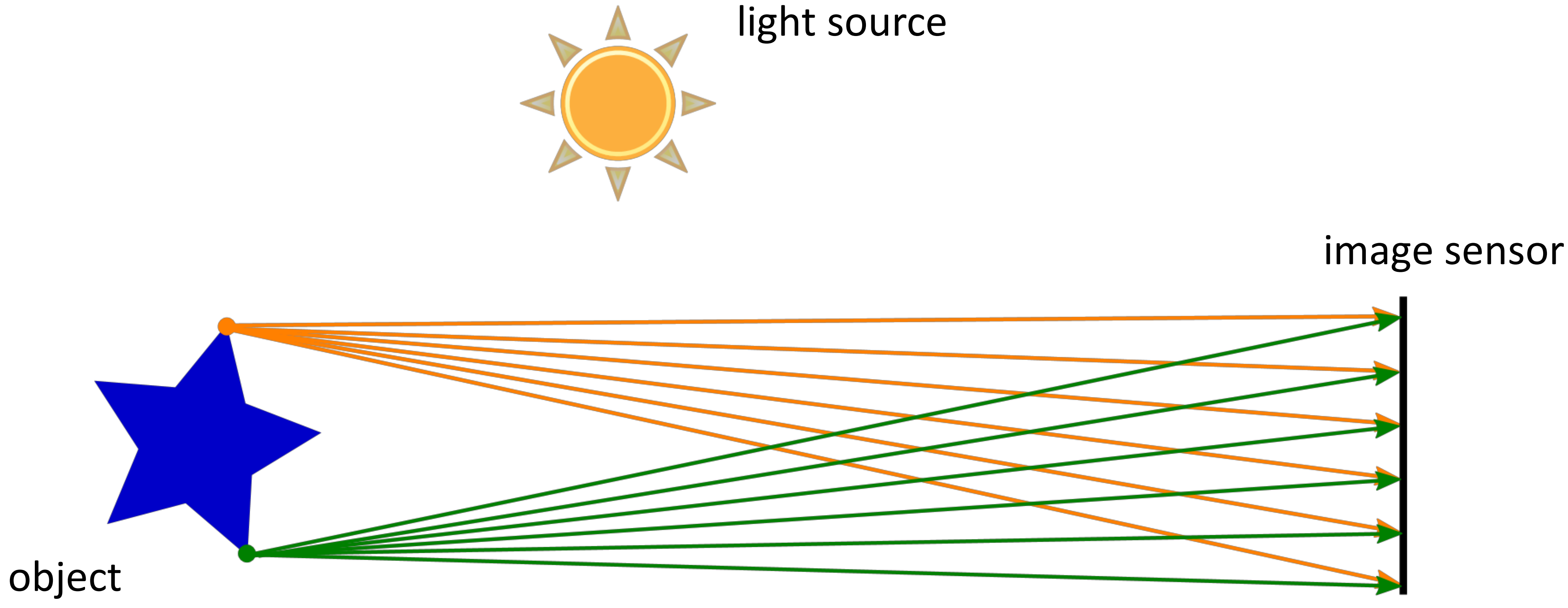
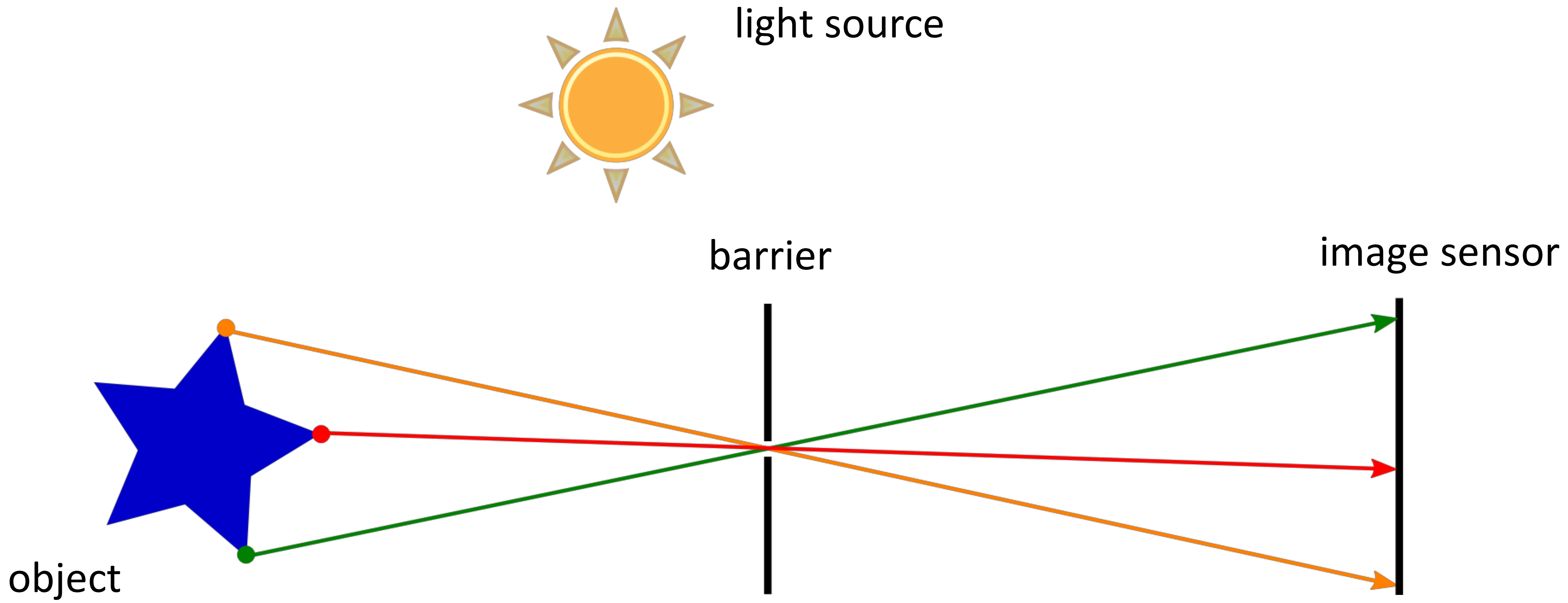
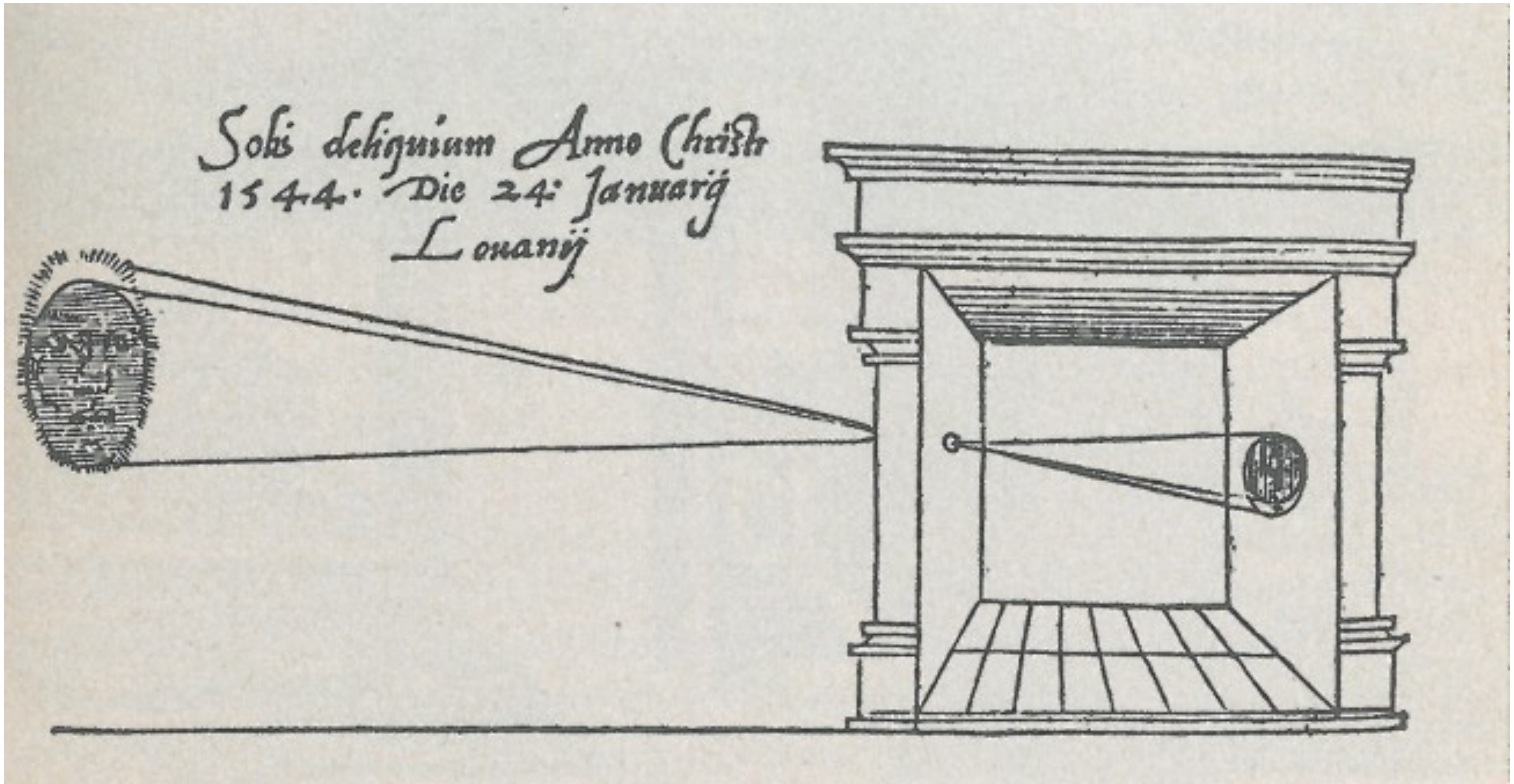


Image Formation



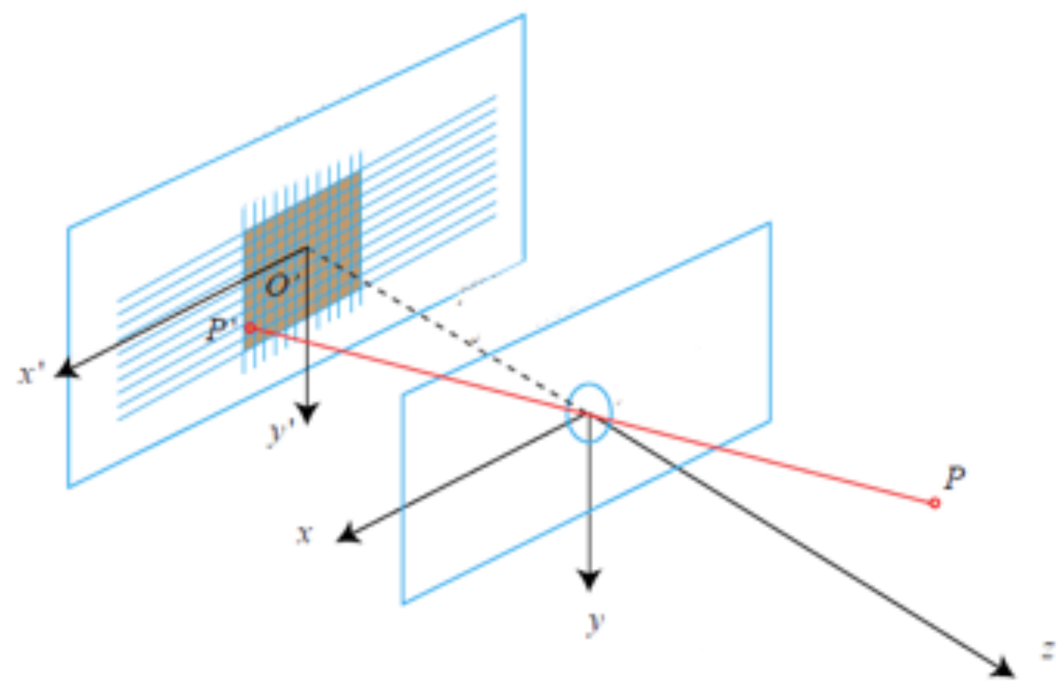
Camera Obscura

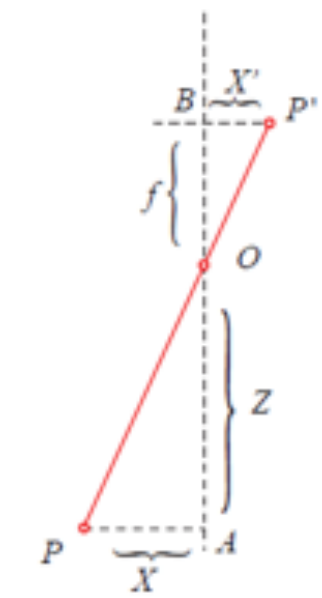


First published picture of camera obscura in Gemma Frisius' 1545 book De Radio Astronomica et Geometrica

Pinhole Camera Model

- Camera coordinate frame attached to the center of (0,0) pixel.
  - X - horizontal axis
  - Y - vertical axis downwards
  - Z - forward





Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y\right]^T$$

Projection:

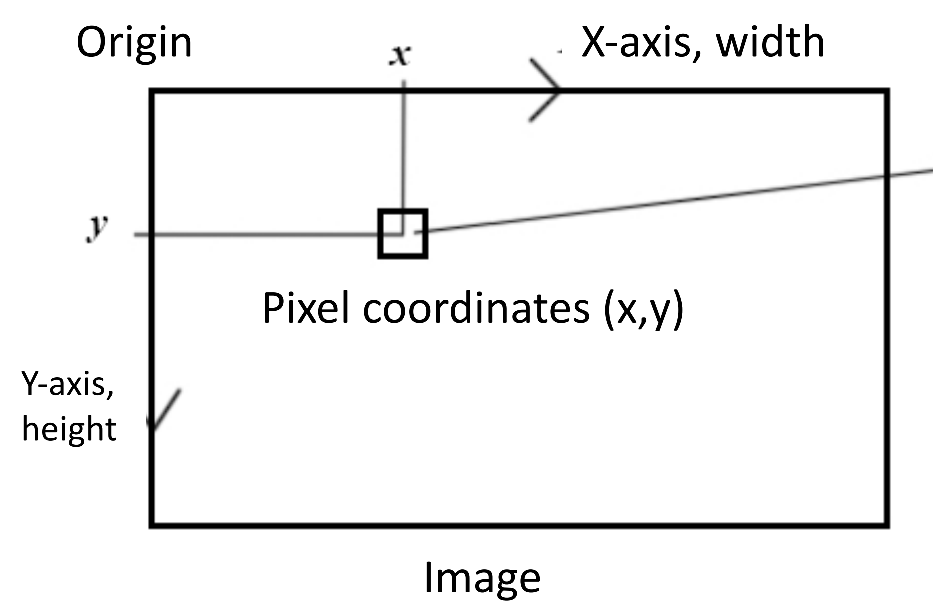
$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \frac{x}{z} \\ f_y \frac{y}{z} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix}$$

Unprojection:

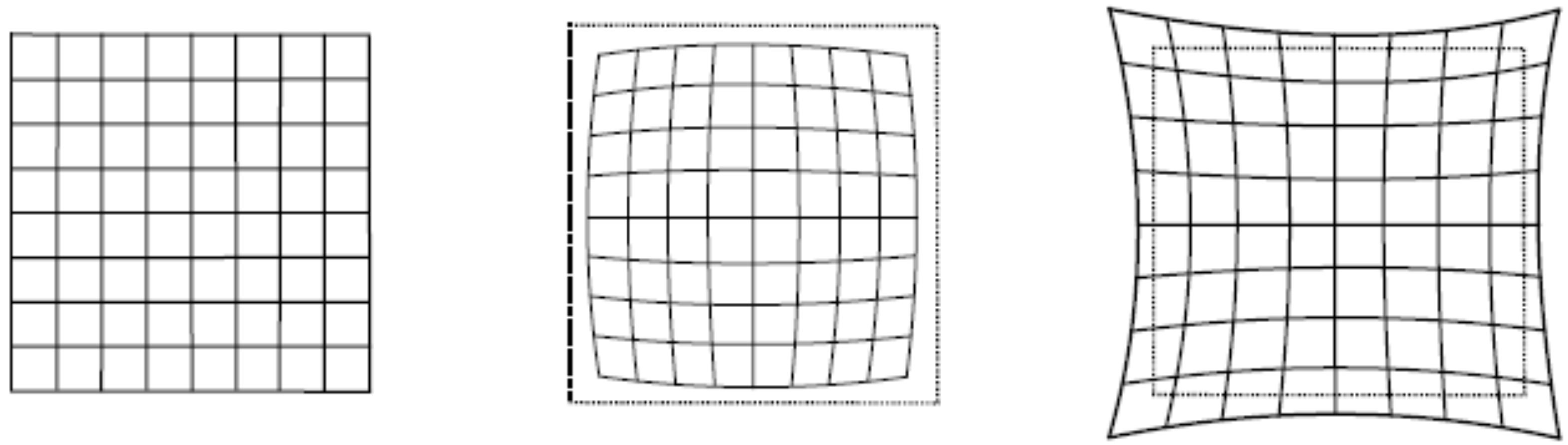
$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \frac{1}{\sqrt{m_x^2 + m_y^2 + 1}} \begin{bmatrix} m_x \\ m_y \\ 1 \end{bmatrix}$$

$$m_x = \frac{u - c_x}{f_x},$$

$$m_y = \frac{v - c_y}{f_y}.$$

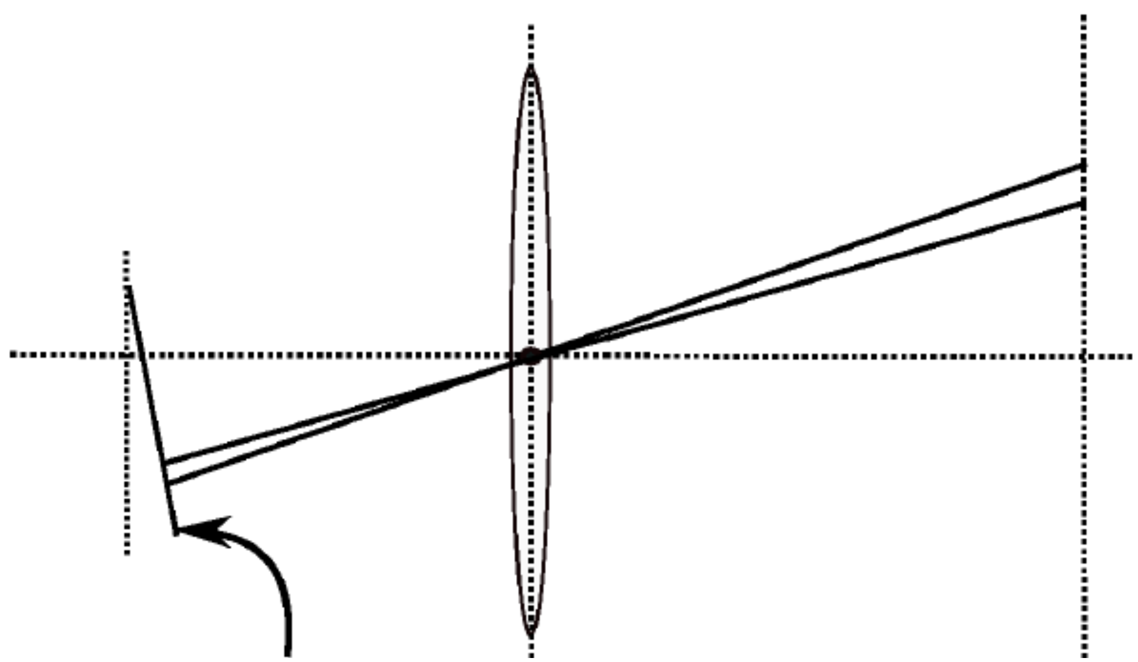


Distortion



Original image Barrel distortion Pincushion distortion

Image plane



Large FOV and Navigation

Z. Zhang, H. Rebecq, C. Forster, D. Scaramuzza “**Benefit of Large Field-of-View Cameras for Visual Odometry**”
IEEE International Conference on Robotics and Automation (ICRA), Stockholm, 2016.

Distortion

Pinhole-Undistorted

- Pinhole
  - Fast projection and unprojection
  - Not suitable for > 180°
  - Bad numeric properties > 120°

Original Image

- More complex model
  - Working with “raw” image
  - No issues with large FOV
  - Possible to optimize intrinsics online

(Extended) Unified Camera Model

Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y, \alpha, \beta\right]^T$$

Projection:

$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \frac{x}{\alpha d + (1-\alpha)z} \\ f_y \frac{y}{\alpha d + (1-\alpha)z} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix},$$

$$d = \sqrt{\beta(x^2 + y^2) + z^2}.$$

Unprojection:

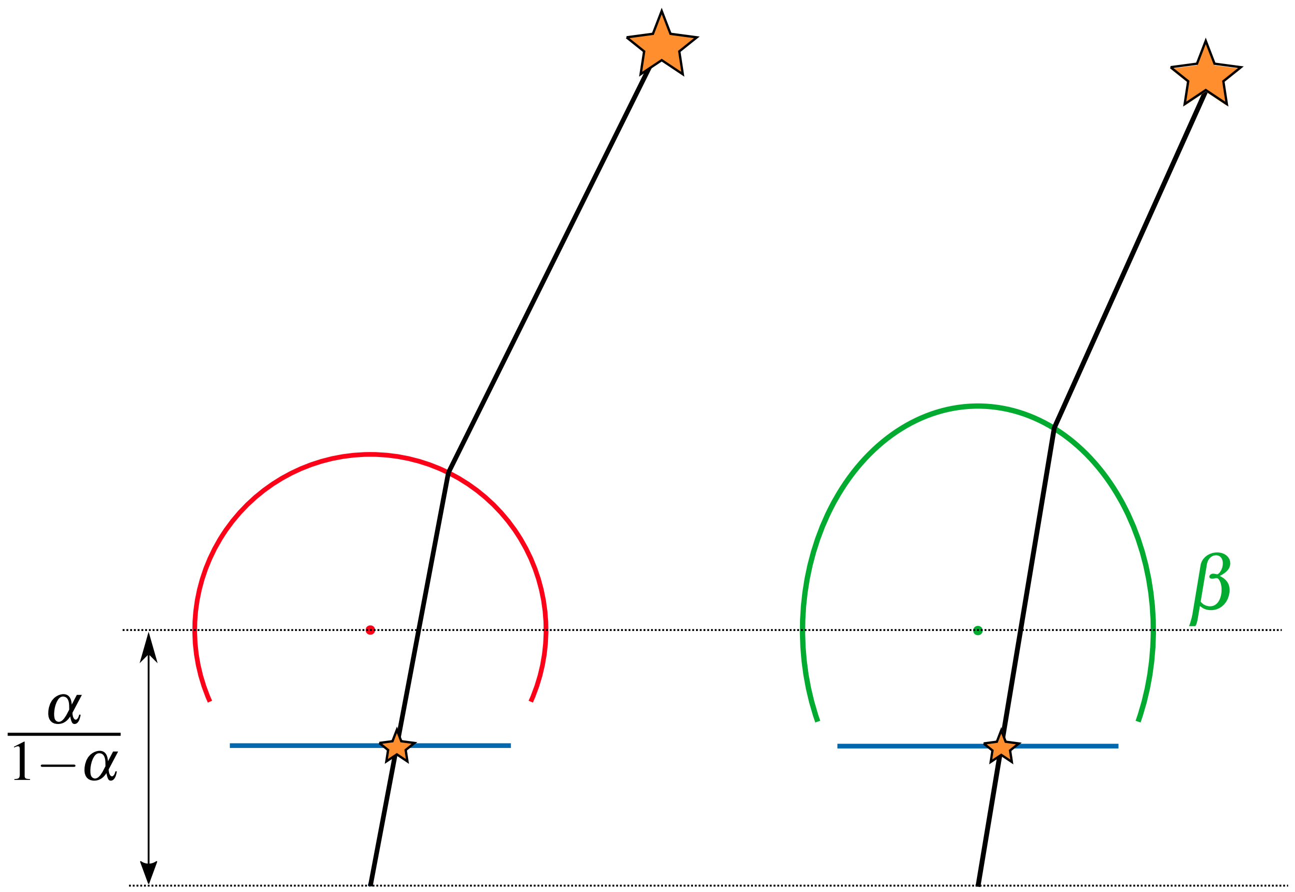
$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \frac{1}{\sqrt{m_x^2 + m_y^2 + m_z^2}} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix},$$

$$m_x = \frac{u - c_x}{f_x},$$

$$m_y = \frac{v - c_y}{f_y},$$

$$r^2 = m_x^2 + m_y^2,$$

$$m_z = \frac{1 - \beta\alpha^2 r^2}{\alpha\sqrt{1 - (2\alpha - 1)\beta r^2} + (1 - \alpha)},$$



Kannala-Brandt Camera Model

Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y, k_1, k_2, k_3, k_4\right]^T$$

Projection:

$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \, d(\theta) \, \frac{x}{r} \\ f_y \, d(\theta) \, \frac{y}{r} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix},$$

$$r = \sqrt{x^2 + y^2},$$

$$\theta = \text{atan2}(r, z),$$

$$d(\theta) = \theta + k_1\theta^3 + k_2\theta^5 + k_3\theta^7 + k_4\theta^9 .$$

Unprojection:

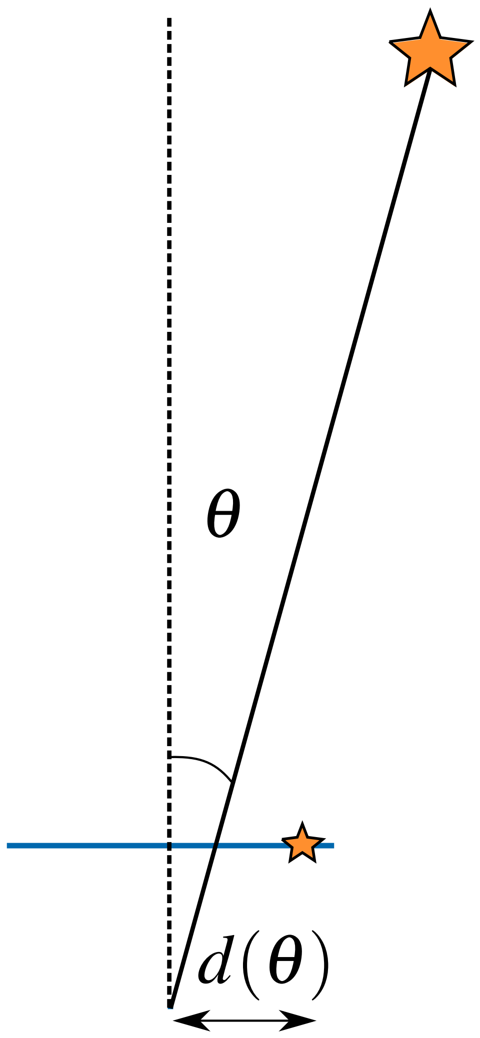
$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \begin{bmatrix} \sin(\theta^*) \, \frac{m_x}{r_u} \\ \sin(\theta^*) \, \frac{m_y}{r_u} \\ \cos(\theta^*) \end{bmatrix},$$

$$m_x = \frac{u - c_x}{f_x},$$

$$m_y = \frac{v - c_y}{f_y},$$

$$r_u = \sqrt{m_x^2 + m_y^2},$$

$$\theta^* = d^{-1}(r_u),$$



Double Sphere Camera Model

Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y, \xi, \alpha\right]^T$$

Projection:

$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \frac{x}{\alpha d_2 + (1-\alpha)(\xi d_1 + z)} \\ f_y \frac{y}{\alpha d_2 + (1-\alpha)(\xi d_1 + z)} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix},$$

$$d_1 = \sqrt{x^2 + y^2 + z^2},$$

$$d_2 = \sqrt{x^2 + y^2 + (\xi d_1 + z)^2}.$$

Unprojection:

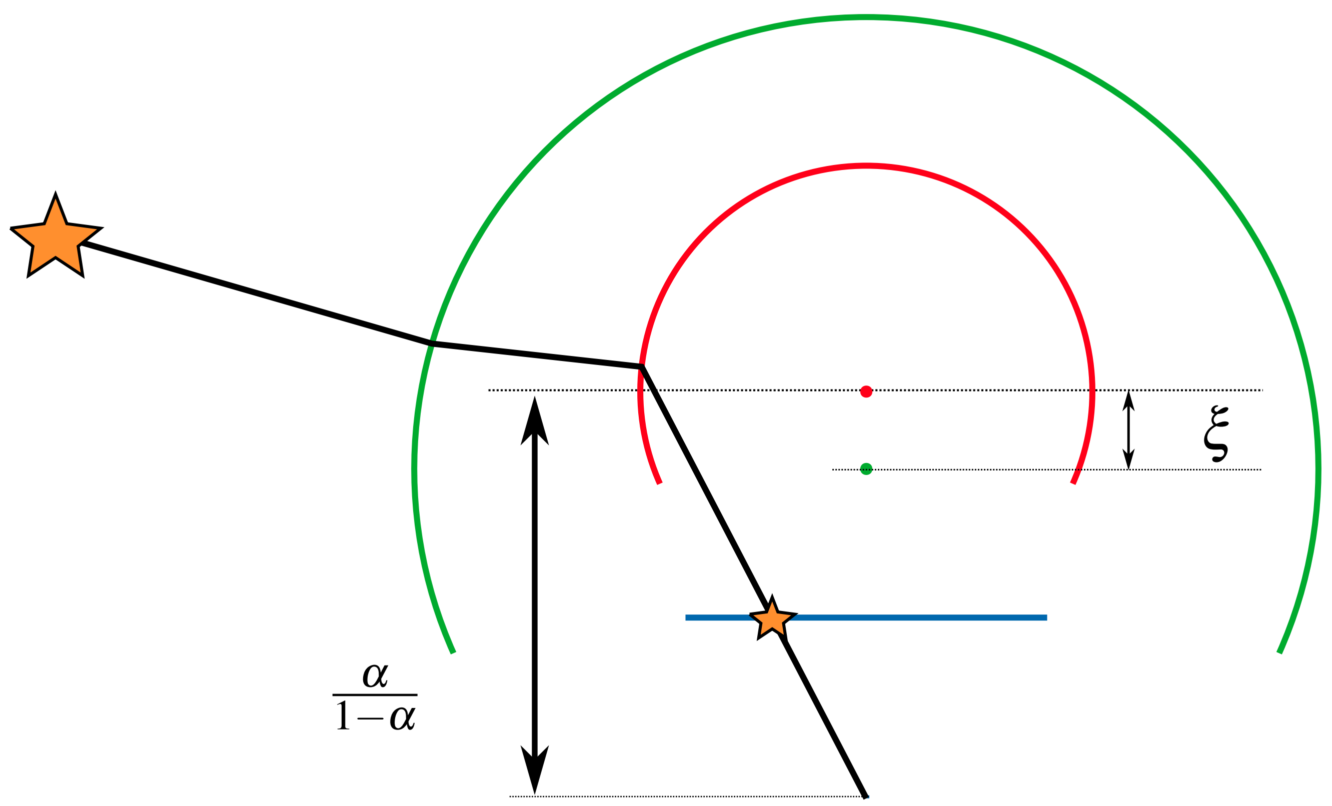
$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \frac{m_z \xi + \sqrt{m_z^2 + (1-\xi^2) r^2}}{m_z^2 + r^2} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \xi \end{bmatrix},$$

$$m_x = \frac{u - c_x}{f_x},$$

$$m_y = \frac{v - c_y}{f_y},$$

$$r^2 = m_x^2 + m_y^2,$$

$$m_z = \frac{1 - \alpha^2 r^2}{\alpha \sqrt{1 - (2\alpha - 1) r^2} + 1 - \alpha}.$$



The Double Sphere Camera Model (V. Usenko, N. Demmel and D. Cremers), *In Proc. of the Int. Conference on 3D Vision (3DV)*, 2018. [arXiv:1807.08957]

Camera Models Code

```
template <typename Scalar>
class PinholeCamera : public AbstractCamera<Scalar> {
 public:
  …
  typedef Eigen::Matrix<Scalar, 2, 1> Vec2;
  typedef Eigen::Matrix<Scalar, 3, 1> Vec3;
  typedef Eigen::Matrix<Scalar, N, 1> VecN;

  PinholeCamera() { param.setZero(); }
  PinholeCamera(const VecN& p) { param = p; }
  …
  virtual Vec2 project(const Vec3& p) const {
    const Scalar& fx = param[0];
    const Scalar& fy = param[1];
    const Scalar& cx = param[2];
    const Scalar& cy = param[3];

    const Scalar& x = p[0];
    const Scalar& y = p[1];
    const Scalar& z = p[2];

    Vec2 res;
    // TODO SHEET 2: implement camera model
    return res;
  }

  virtual Vec3 unproject(const Vec2& p) const {
    const Scalar& fx = param[0];
    const Scalar& fy = param[1];
    const Scalar& cx = param[2];
    const Scalar& cy = param[3];

    Vec3 res;
   // TODO SHEET 2: implement camera model
    return res;
  }
  …
  EIGEN_MAKE_ALIGNED_OPERATOR_NEW
 private:
  VecN param;
};
```

- Avoid using std::pow() function to maintain the precision. For example, if you need to compute x^2 use multiplication: $x * x$.
- If your compiler complains about Jet types try changing the constants in projection and unprojection functions to Scalar(<constant>). For example, Scalar(1) instead of 1.
- You can use Newton's method for finding roots to compute a root of the polynomial given a good initialization. Usually 3-5 iterations should be enough for the optimization to converge.
- You can use Horner's method to efficiently compute polynomials.

Optimization

Maximum a Posteriori Estimation

Given a set of parameters $x = \{x_1, \ldots x_n\}$ and a set of observations that depend on the parameters $z = \{z_1, \ldots z_m\}$ we want estimate the value of x that is most likely to result in these observations:

$$x^* = \underset{x}{\text{argmax}}\ P(x\,|\,z),$$

This estimate of the parameters x^* is called the Maximum a posteriori (MAP) estimation.

We can rewrite the probability using the Bayes' Rule:

Posteriori Likelihood Prior

$$P(x\,|\,z) = \frac{P(z\,|\,x)P(x)}{P(z)}.$$

We can drop the denominator, because it does not depend on x.

$$x^* = \underset{x}{\text{argmax}}\ P(z\,|\,x)P(x).$$

"Which state it is most likely to produce such measurements?"

From MAP to Least Squares

- From MAP to least squares problem
- If we assume that the measurements are independent the joint PDF can be factorized:

$$P(z \mid x) = \prod_{k=0}^{K} P(z_k \mid x)$$

- Let's consider a single observation: $z_k = h(x) + v_k,$
  - Affected by Gaussian noise: $v_k \sim N(0,\ Q_k)$

- The observation model gives us a conditional PDF:

$$P(z_k \mid x) = N(h(x),\ Q_k)$$

- How do we estimate x ?

From MAP to Least Squares

- Gaussian Distribution (matrix form)

$$P(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right).$$

- Take negative logarithm from both sides:

$$-\ln P(x) = \frac{1}{2}\ln((2\pi)^p|\Sigma|) + \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu).$$

- Maximum of $P(x)$ is equivalent to the minimum of $-\ln P(x)$.

From MAP to Least Squares

- Batch least squares
- Formulate residual function:

$$r_k = z_k - h(x).$$

- Maximizing of $P(x)$ is equivalent to the minimizing the sum of squared residuals:

$$E(x) = \frac{1}{2} \sum_k r_k^T Q r_k.$$

From MAP to Least Squares

- Some notes:
  - Because of noise, when we take the estimated trajectory and map into the models, they won't fit perfectly
  - Then we adjust our estimation to get a better estimation (minimize the error)
  - The error distribution is affected by noise distribution (information matrix)
- Structure of the least square problem
  - Sum of many squared errors
  - The dimension of total state variable may be high
  - But single error item is easy (only related to two states in our case)
  - If we use Lie group and Lie algebra, then it's a non-constrained least square

$$E(x) = \frac{1}{2} \sum_k r_k^T Q r_k$$

Least Squares

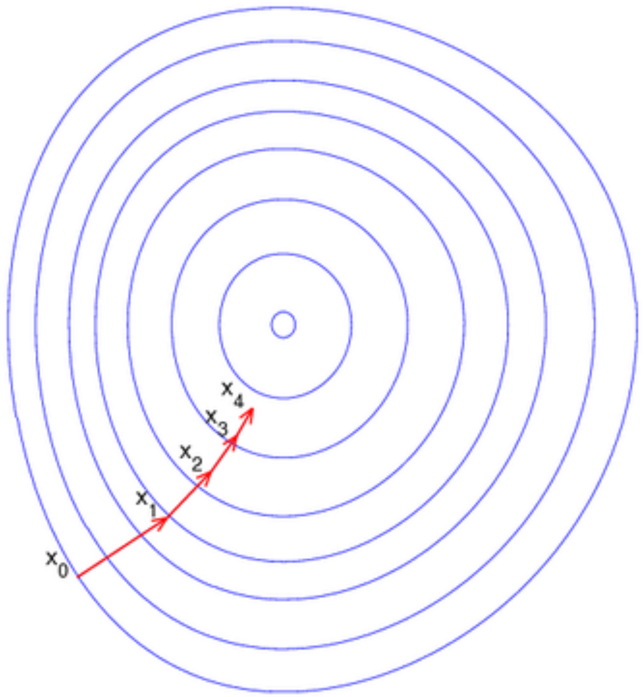
- How to solve a least square problem?
  - Non-linear, discrete time, non-constrained
- Let's start from a simple example
  - Consider minimizing a squared error: $$E(x) = \frac{1}{2}\sum_k r_k(x)^T r_k(x) = \frac{1}{2} r(x)^T r(x)$$
  - When $E(x)$ is simple, just solve:

$$\frac{\partial E(x)}{\partial x} = 0$$

- And we will find the maxima/minima/saddle points

Least Squares

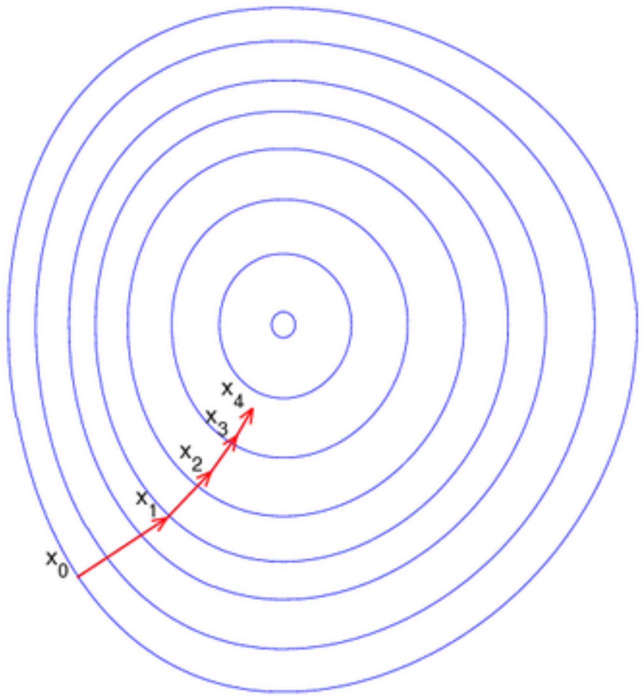
- When $E(x)$ is a complicated function:
  - $\frac{\partial E(x)}{\partial x} = 0$ is hard to solve
  - We use iterative methods



- Iterative methods
  1. Start from an initial estimate x_0
  2. At iteration n, we find an increment Δx_n that minimizes $E(x_n + \Delta x_n)$.
  3. If the change in error function is small enough, stop (converged).
  4. If not, set $x_{n+1} = x_n + \Delta x_n$ and iterate to step 2.

Gradient Descent

- How to find the increment?
- First order methods - Gradient Descent
  - Taylor expansion of the objective function
  - $E(x + \Delta x) = E(x) + G(x)\Delta x$

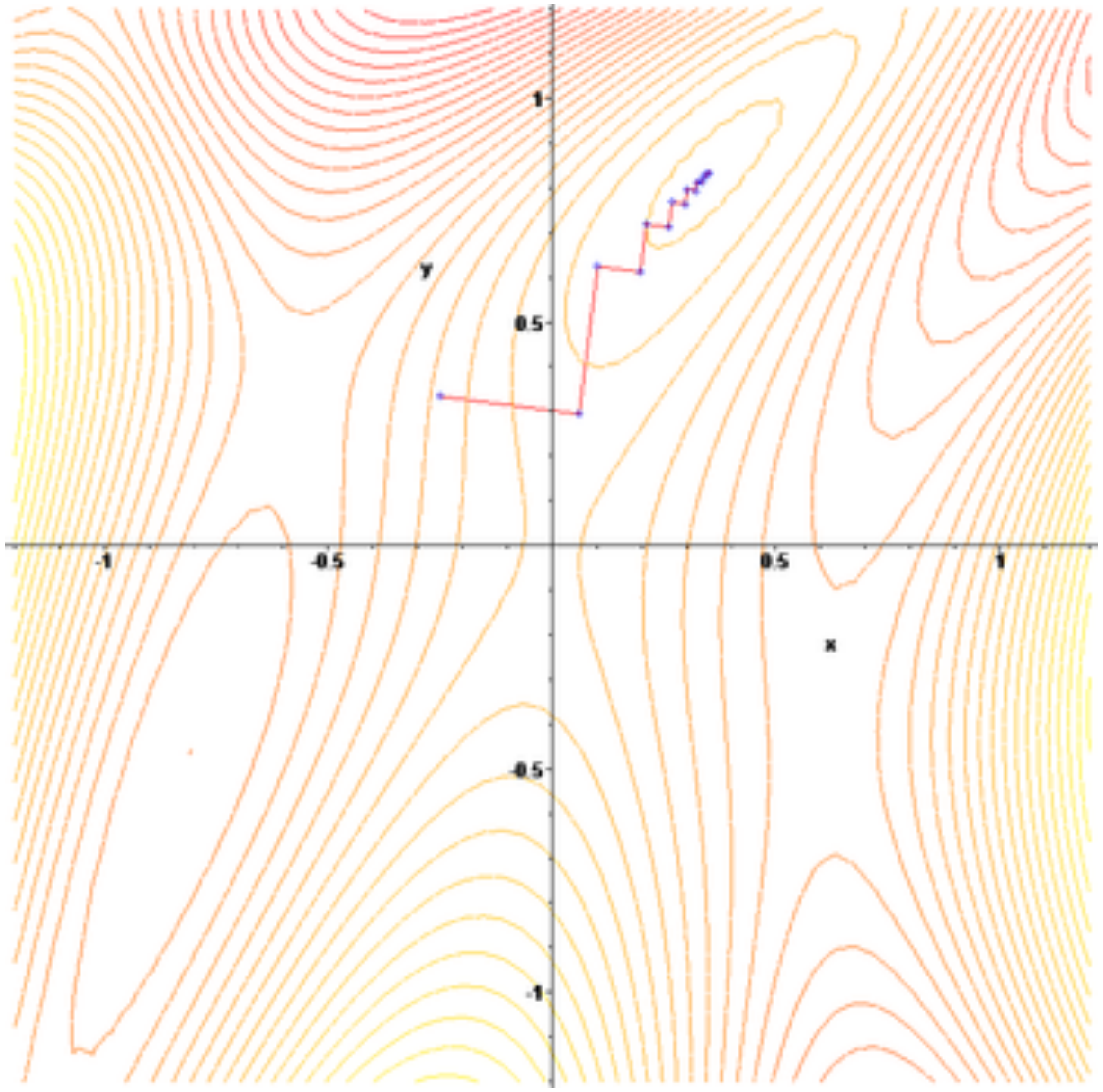


The update step:

$\Delta x = -\alpha G(x)$

Gradient Descent Performance

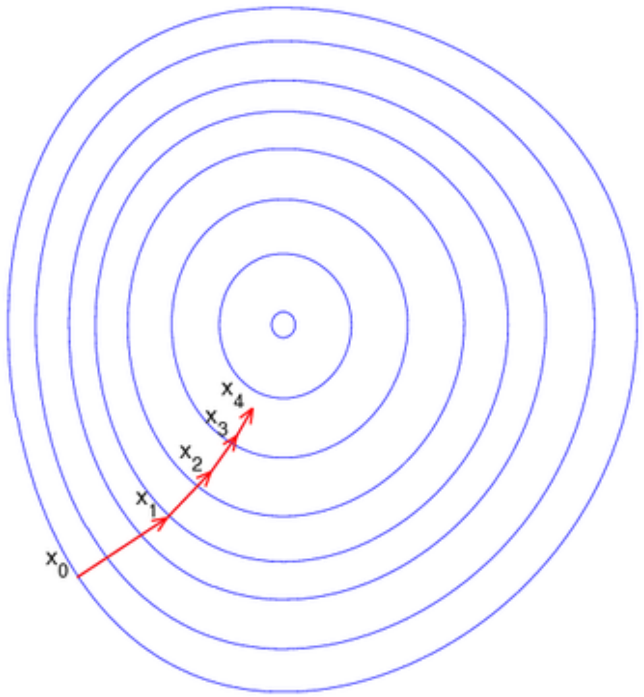
Zig-zag in steepest descent:



- Other shortcomings:
  - Slow convergence speed
  - Even slower when close to minimum

Second Order Methods

- Second order methods
  - Taylor expansion of the objective function
  - $E(x + \Delta x) = E(x) + G(x)\Delta x + \Delta x^T H(x)\Delta x$
  - Setting $\dfrac{\partial E(x + \Delta x)}{\partial \Delta x} = 0$



The update step:

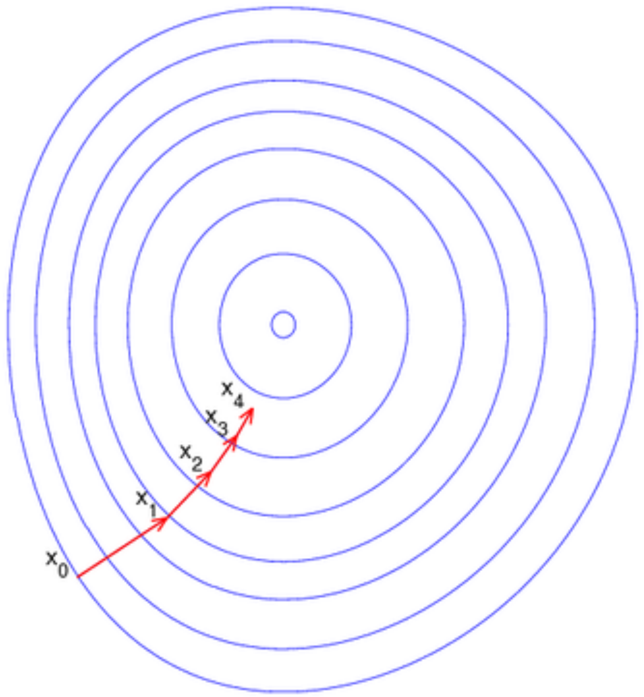
$$H(x)\Delta x = -G(x) \implies \Delta x = -H^{-1}(x)\ G(x)$$

This is called Newton's method.

Second Order Methods

- Second order method converges more quickly than first order methods
- But the Hessian matrix may be hard to compute

- Can we avoid the Hessian matrix and also keep second order's convergence speed?
  - Gauss-Newton
  - Levenberg-Marquardt



Gauss-Newton Method

- Gauss-Newton
  - Taylor expansion of $r(x)$: $r(x + \Delta x) \simeq r(x) + J(x)\Delta x$
  - Then the squared error becomes:

$$E(x + \Delta x) = \frac{1}{2} r(x)^T r(x) + \Delta x^T J(x)^T r(x) + \frac{1}{2} \Delta x^T J(x)^T J(x) \Delta x$$

$$= F(x) + \Delta x^T J(x)^T r(x) + \frac{1}{2} \Delta x^T J(x)^T J(x) \Delta x$$

If we set $\frac{\partial E(x + \Delta x)}{\partial \Delta x} = 0$ we get:

$$J^T(x) J(x) \Delta x = - J(x)^T r(x) \implies \Delta x = - \left(J^T(x) J(x)\right)^{-1} J(x)^T r(x)$$

$\simeq H(x)$ Newton's Method $\simeq G(x)$

Gauss-Newton Method

- Gauss-Newton uses $J^T(x)J(x)$ as an approximation of the Hessian
  - Avoids the computation of $H(x)$ in the Newton's method

- But $J^T(x)J(x)$ is only semi-positive definite
- $H(x)$ maybe singular when $J^T(x)J(x)$ has null space

Levenberg-Marquardt Method

- Trust region approach: approximation is only valid in a region
- Evaluate if the approximation is good:

$$\rho = \frac{r(x + \Delta x) - r(x)}{J(x)\Delta x}.$$

Real descent/approx. descent

- If ρ is large, increase the region
- If ρ is small, decrease the region

- LM optimization:

$$E(x + \Delta x) = \frac{1}{2} r(x + \Delta x)^T r(x + \Delta x) + \lambda \|\Delta x\|^2$$

  - Assume the approximation is only good within a region
  - λ controls the region based on ρ

Levenberg-Marquardt Method

- Trust region problem:

$$E(x + \Delta x) = \frac{1}{2} r(x + \Delta x)^T r(x + \Delta x) + \lambda \|\Delta x\|^2$$

- Expand it just like in GN case, the incremental is:

$$\Delta x = -\left(J^T(x) J(x) + \lambda I\right)^{-1} J(x)^T r(x)$$

- The λI part makes sure that Hessian is positive definite.
- When $\lambda = 0$ LM becomes GN.
- When $\lambda \rightarrow \infty$ LM becomes gradient descent.

Other Methods

- Dog-leg method
- Conjugate gradient method
- Quasi-Newton’s method
- Pseudo-Newton’s method
- …

- You can find more in optimization books if you are interested

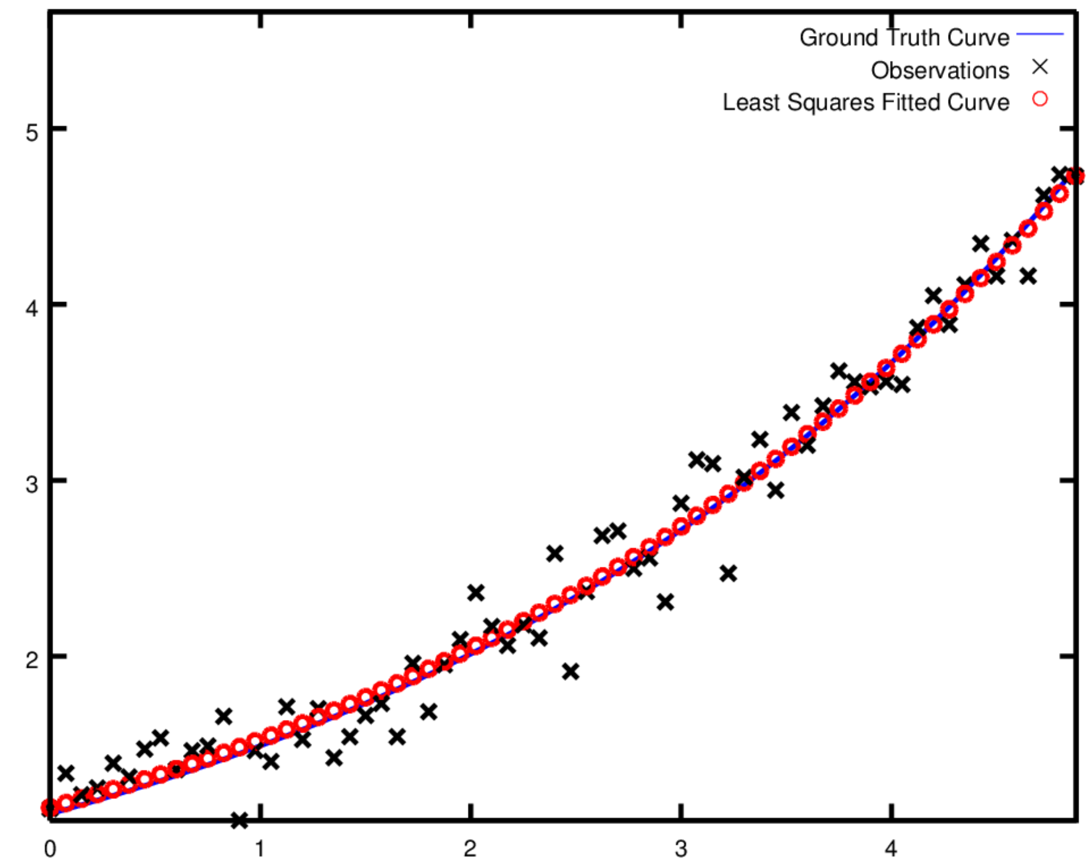
- In SLAM/SfM/VO, Gauss-Netwton and Levenberg-Marquardt are used to solve camera motion, optical-flow, etc.

More details:

Triggs B, McLauchlan PF, Hartley RI, Fitzgibbon AW. Bundle adjustment—a modern synthesis. InInternational workshop on vision algorithms 1999 Sep 20 (pp. 298-372). Springer, Berlin, Heidelberg.

Ceres

- We will use Ceres for least-squares optimization.
- Tutorial: http://ceres-solver.org/tutorial.html
- Curve fitting example: $y = \exp(mx + c)$
- Observations: a set of (x, y) pairs
- Parameters to estimate: m, c.



Ceres

- Define your residual class as a functor (overload the () operator)

```
struct ExponentialResidual {
  ExponentialResidual(double x, double y)
      : x_(x), y_(y) {}

  template <typename T>
  bool operator()(const T* const m, const T* const c, T*
residual) const {
    residual[0] = T(y_) - exp(m[0] * T(x_) + c[0]);
    return true;
  }

 private:
  // Observations for a sample.
  const double x_;
  const double y_;
};
```

Ceres

- Build the optimization problem

```
double m = 0.0;
double c = 0.0;

Problem problem;
for (int i = 0; i < kNumObservations; ++i) {
  CostFunction* cost_function =
       new AutoDiffCostFunction<ExponentialResidual, 1, 1, 1>(
           new ExponentialResidual(data[2 * i], data[2 * i + 1]));
  problem.AddResidualBlock(cost_function, NULL, &m, &c);
}
```

- With auto-diff, Ceres will compute the Jacobians for you

Ceres

- Finally solve it by calling the Solve() function and get the result summary
- You can set some parameters like number of iterations, stop conditions or the linear solver type.

```
// Run the solver!
 Solver::Options options;
 options.linear_solver_type = ceres::DENSE_QR;
 options.minimizer_progress_to_stdout = true;
 Solver::Summary summary;
 Solve(options, &problem, &summary);

 std::cout << summary.BriefReport() << "\n";
 std::cout << "m : " << m
           << “c : " << c << "\n";
```

Least-Squares Summary

- In the maximum a posteriori estimation we estimate all the state variable given using a set of noisy measurements.
- The MAP estimation problem with Gaussian noise can be reformulated into a least square problem
- It can be solved by iterative methods: Gradient Descent, Newton’s method, Gauss-Newton or Levernberg-Marquardt.

Exercise 2

- We want to estimate
  - Poses of the camera setup with respect to pattern
  - Intrinsic parameters of both cameras
  - Extrinsic parameters (rigid body transformation from one camera to the other)

- Minimizing the projection residuals:

$$r = u_j - \pi(R_{cw} p_w^j + t_{cw}, \mathbf{i}),$$

- u_j - detection of the corner j in the image.
- p_w^j - 3D coordinates in the world (pattern) coordinate frame
- $\mathbf{i}$ - intrinsic parameter of the camera
- R_{cw}, t_{cw} - rigid body transformation from the world (pattern) coordinate frame to the camera coordinate frame.
- π - is the projection function

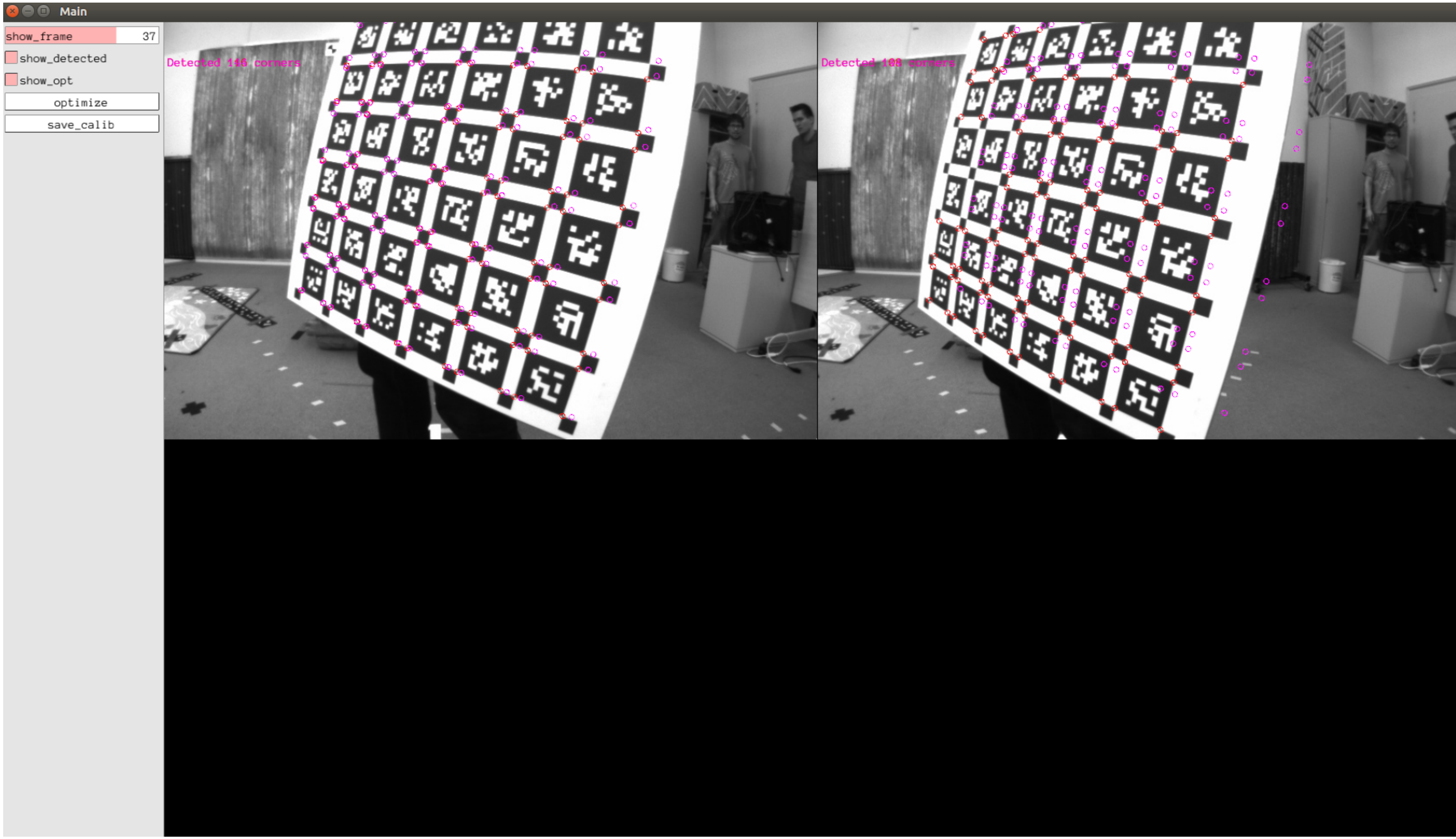
- Corner points are detected using Apriltags

E. Olson. AprilTag: A robust and flexible visual fiducial system. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA)*, pages 3400–3407. IEEE, May 2011.

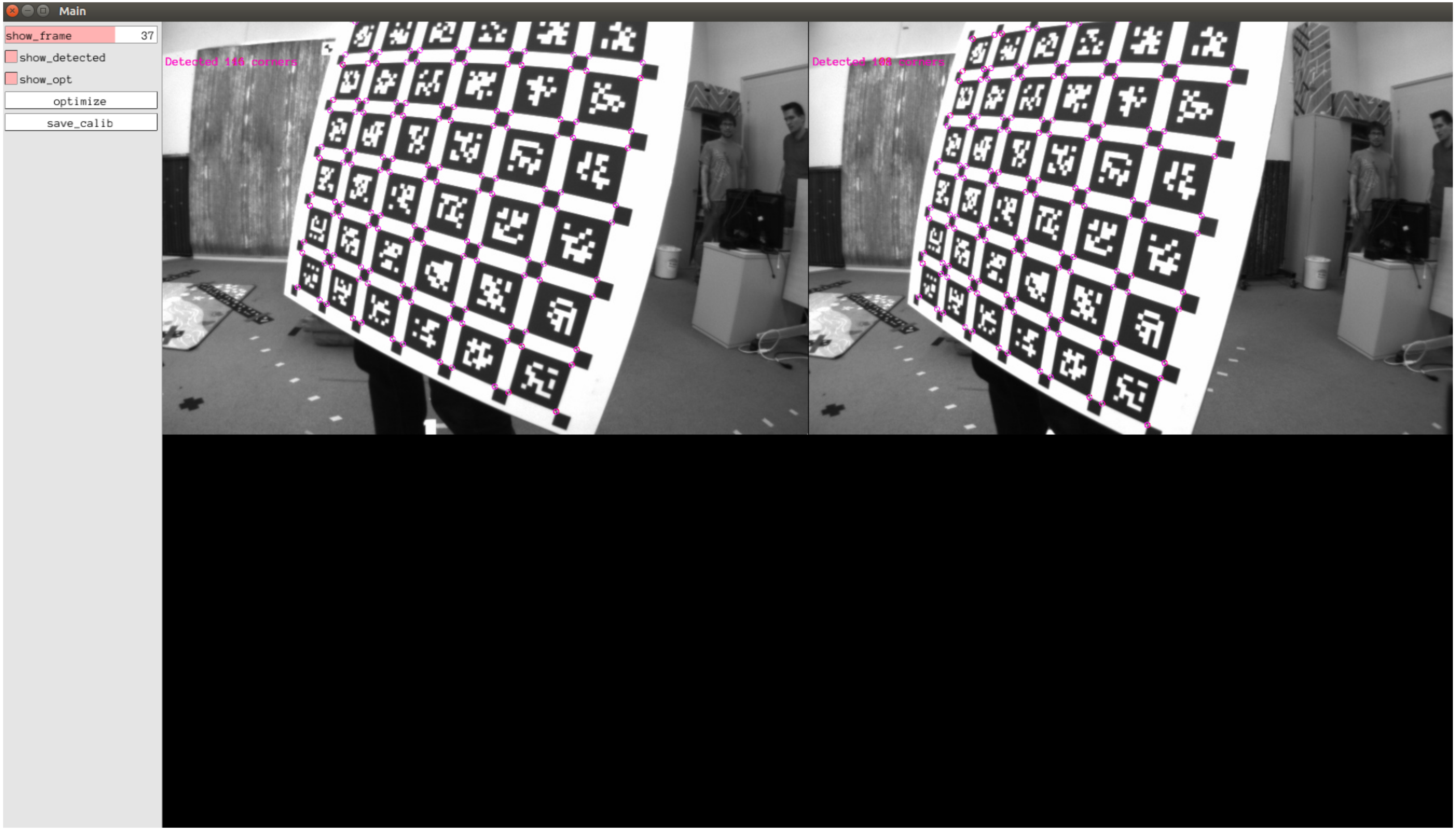
Exercise 2 Residual

```
struct ReprojectionCostFunctor {
  EIGEN_MAKE_ALIGNED_OPERATOR_NEW
  ReprojectionCostFunctor(const Eigen::Vector2d& p_2d,
                          const Eigen::Vector3d& p_3d,
                          const std::string& cam_model)
      : p_2d(p_2d), p_3d(p_3d), cam_model(cam_model) {}

  template <class T>
  bool operator()(T const* const sT_w_i, T const* const sT_i_c,
                  T const* const sIntr, T* sResiduals) const {
    Eigen::Map<Sophus::SE3<T> const> const T_w_i(sT_w_i);
    Eigen::Map<Sophus::SE3<T> const> const T_i_c(sT_i_c);

    Eigen::Map<Eigen::Matrix<T, 2, 1>> residuals(sResiduals);
    const std::shared_ptr<AbstractCamera<T>> cam =
        AbstractCamera<T>::from_data(cam_model, sIntr);

    // TODO SHEET 2: implement the rest of the functor
    return true;
  }

  Eigen::Vector2d p_2d;
  Eigen::Vector3d p_3d;
  std::string cam_model;
};
```

Exercise 2


Main
show_frame
37
show_detected
show_opt
optimize
save_calib

Exercise 2


Main
show_frame
37
show_detected
show_opt
optimize
save_calib
Detected 146 corners
Detected 108 corners

Exercise 2

- Use camera models presented here to get initial projections
- Implement the projection function
- Implement the residual.
- Set up optimization problem. Use local parametrization where necessary.

- Test different models. How well do they fit the lens?