Multiple View Geometry: Exercise Sheet 2

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Part I: Theory

1. Which groups have you seen in the lecture? Write down the names and the correct inclusions! (e.g.: group A $\subset$ group B)

2. Let A be a symmetric matrix, and λ_a, λ_b eigenvalues with eigenvectors v_a and v_b. Prove: if v_a and v_b are not orthogonal, it follows: $\lambda_a = \lambda_b$.

   Hint: What can you say about $\langle Av_a, v_b \rangle$?

3. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with the orthonormal basis of eigenvectors $v_1, \ldots, v_n$ and eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Find all vectors x, that minimize the following term:

   $$\min_{||x||=1} x^\top A x$$

   How many solutions exist? How can the term be maximized?

   Hint: Use the expression $x = \sum_{i=1}^{n} \alpha_i v_i$ with coefficients $\alpha_i \in \mathbb{R}$ and compute appropriate coefficients!

4. Let $A \in \mathbb{R}^{m \times n}$. Prove that kernel$(A) =$ kernel$(A^\top A)$.

   Hint: Consider a) $x \in$ kernel$(A) \Rightarrow x \in$ kernel$(A^\top A)$
   and b) $x \in$ kernel$(A^\top A) \Rightarrow x \in$ kernel(A).

5. Singular Value Decomposition (SVD)

   Let $A = USV^\top$ be the SVD of A.

   (a) Write down possible dimensions for A, U, S and V.

   (b) What are the similarities and differences between the SVD and the eigenvalue decomposition?

   (c) What do you know about the relationship between U, S, V and the eigenvalues and eigenvectors of $A^\top A$ and $AA^\top$?

   (d) What is the interpretation of the entries in S and what do the entries of S tell us about A?

Part II: Practical Exercises

The Moore-Penrose pseudo-inverse

To solve the linear system $Ax = b$ for an arbitrary (non-quadratic) matrix $A \in \mathbb{R}^{m\times n}$ of rank $r \leq \min(m,n)$, one can define a (generalized) inverse, also called the *Moore-Penrose pseudo-inverse* (refer to Chapter 1, last slide).
In this exercise we want to solve the linear system $Dx = b$ with $D \in \mathbb{R}^{m\times 4}$, $b \in \mathbb{R}^m$ a vector whose components are all equal to 1, and $x^* = [4, -3, 2, -1]^T \in \mathbb{R}^4$ should be one possible solution of the linear system, i.e. for any row $[d_1, d_2, d_3, d_4]$ of D:

$$4d_1 - 3d_2 + 2d_3 - d_4 = 1$$

We recall that the set of all possible solutions is given by $S = \{x^* + v \mid v \in \text{kernel}(D)\}$.

1. Create some data
   (a) Generate such a matrix D using random values with $m = 4$ rows.
   (Hint: Use `rand` to define d_1, d_2, d_3 and set $d_4 = 4d_1 - 3d_2 + 2d_3 - 1$.)
   In general, $rank(D) = 4$, hence there is a unique solution.
   (b) Introduce small additive errors into the data.
   (Hint: Use `eps*rand` with `eps=1.e-4`)
2. Find the coefficients x solving the system $Dx = b$
   (a) Compute the SVD for the matrix D.
   (Hint: Use `svd`)
   (b) Compute the Moore-Penrose pseudo-inverse using the result from (a), and compare it to the output of the `MATLAB` function `pinv`.
   (c) Compute the coefficients x, and compare it to the true solution x^*.
3. Repeat the two previous questions, by setting m to a higher value. How is the precision impacted?
4. We assume in the following that $m = 3$, hence we have infinitely many solutions.
   (a) Solve again the linear system using questions (1) and (2).
   Thus $rank(D) = 3$ and dim(kernel(D)) $= 1$.
   (b) Use the function `null` to get a vector $v \in$ kernel(D).
   The set of all possible solutions is $S = \{x + \lambda v \mid \lambda \in \mathbb{R}\}$.
   (c) According to the last slide of Chapter 1, we know that the following statement holds:
   $x_{min} = A^+ b$ *is among all minimizers of* $\|Ax - b\|^2$ *the one with the smallest norm* $|x|$.

   Let $\lambda \in \mathbb{R}$, $x_\lambda = x + \lambda v$ one possible solution, and $e_\lambda = \|Dx_\lambda - b\|^2$ the associated error.
   Using the function `plot`, display both graphs of $\|x_\lambda\|$ and e_λ according to $\lambda \in \{-100, \ldots, 100\}$, and observe that the statement indeed holds.