Multiple View Geometry: Exercise Sheet 10

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Part I: Theory

1. **Variational Calculus and Euler-Lagrange**

(a) Under the assumption that h vanishes at the boundary of Ω, prove that

$$\frac{\mathrm{d}E(u)}{\mathrm{d}u} = \frac{\partial \mathcal{L}(u, \nabla u)}{\partial u} - \operatorname{div}\left(\frac{\partial \mathcal{L}(u, \nabla u)}{\partial(\nabla u)}\right) .$$

We can expand $\mathcal{L}(u + \epsilon h, \nabla u + \epsilon \nabla h)$ in terms of ϵ:

$$\mathcal{L}(u + \epsilon h, \nabla u + \epsilon \nabla h) = \mathcal{L}(u, \nabla u) + \epsilon h \frac{\partial \mathcal{L}}{\partial u} + \epsilon \nabla h \frac{\partial \mathcal{L}}{\partial(\nabla u)} + \mathcal{O}(\epsilon^2)$$

Inserting into $\left.\frac{\delta E(u)}{\delta u}\right|_h$ gives

$$\left.\frac{\delta E(u)}{\delta u}\right|_h = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega \left(\epsilon h(x) \left.\frac{\partial \mathcal{L}}{\partial u}\right|_{u(x)} + \epsilon \nabla h(x) \left.\frac{\partial \mathcal{L}}{\partial(\nabla u)}\right|_{\nabla u(x)} + \mathcal{O}(\epsilon^2) \right) \mathrm{d}x$$

The ϵ in the first two terms cancels, and in the $\mathcal{O}(\epsilon^2)$ it will go to zero for $\epsilon \to 0$. Integration by parts of the second term yields

$$\int_\Omega \nabla h(x) \frac{\partial \mathcal{L}}{\partial(\nabla u)} \mathrm{d}x = \int_{\partial\Omega} h(x) \frac{\partial \mathcal{L}}{\partial(\nabla u)} \mathrm{ds} - \int_\Omega h(x) \operatorname{div}\left(\frac{\partial \mathcal{L}}{\partial(\nabla u)}\right) \mathrm{d}x =$$

$$= - \int_\Omega h(x) \operatorname{div}\left(\frac{\partial \mathcal{L}}{\partial(\nabla u)}\right) \mathrm{d}x .$$

Thus,

$$\left.\frac{\delta E(u)}{\delta u}\right|_h = \int_\Omega h(x) \left(\frac{\partial \mathcal{L}}{\partial u} - \operatorname{div}\left(\frac{\partial \mathcal{L}}{\partial(\nabla u)}\right) \right) \mathrm{d}x \quad \Rightarrow \quad \text{claim} .$$

(b) Which condition must hold true for a minimizer u_0 of $E(u)$ ...

- ... in general?

$$\frac{\mathrm{d}E(u)}{\mathrm{d}u} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}(u, \nabla u)}{\partial u} = \operatorname{div}\left(\frac{\partial \mathcal{L}(u, \nabla u)}{\partial(\nabla u)}\right) .$$

- ... if $\mathcal{L}(u, \nabla u) = \mathcal{L}(u)$?

$$\frac{\partial \mathcal{L}(u)}{\partial u} = 0 .$$

- ... if $\mathcal{L}(u, \nabla u) = \mathcal{L}(\nabla u)$?

$$\text{div}\left(\frac{\partial \mathcal{L}(\nabla u)}{\partial(\nabla u)}\right) = 0\,.$$

2. **Multiview Reconstruction as Shape Optimization**

(a) Write down the Euler-Lagrange equation for the given energy $E(u)$.
The E-L equations are

$$\frac{\mathrm{d}E(u)}{\mathrm{d}u} = -\text{div}\left(\frac{\partial \mathcal{L}(\nabla u)}{\partial(\nabla u)}\right) = 0 \quad \text{with} \quad \mathcal{L}(\nabla u) = \rho|\nabla u|$$

Taking the derivative w.r.t. ∇u gives

$$0 = -\text{div}\left(\rho(x)\frac{\nabla u(x)}{|\nabla u(x)|}\right)\,.$$

It is also possible (but not neccessarily required) to expand this further using the product rule for divergence:

$$\begin{aligned}\text{div}\left(\rho(x)\frac{\nabla u(x)}{|\nabla u(x)|}\right) &= \nabla\rho(x)^\top\frac{\nabla u(x)}{|\nabla u(x)|} + \rho(x)\text{div}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)\\ &= \nabla\rho(x)^\top\frac{\nabla u(x)}{|\nabla u(x)|} + \rho(x)\frac{\nabla^2 u(x) - 1}{|\nabla u(x)|}\end{aligned}$$

(b) Write down one gradient descent iteration for $E(u)$.

$$u^{(k+1)}(x) = u^{(k)}(x) - \tau\frac{\mathrm{d}E(u)}{\mathrm{d}u} = u^{(k)}(x) + \tau\left(\nabla\rho(x)^\top\frac{\nabla u(x)}{|\nabla u(x)|} + \rho(x)\frac{\nabla^2 u(x) - 1}{|\nabla u(x)|}\right)$$