Practical Course: Vision Based Navigation

Lecture 2: Camera Models and Optimization

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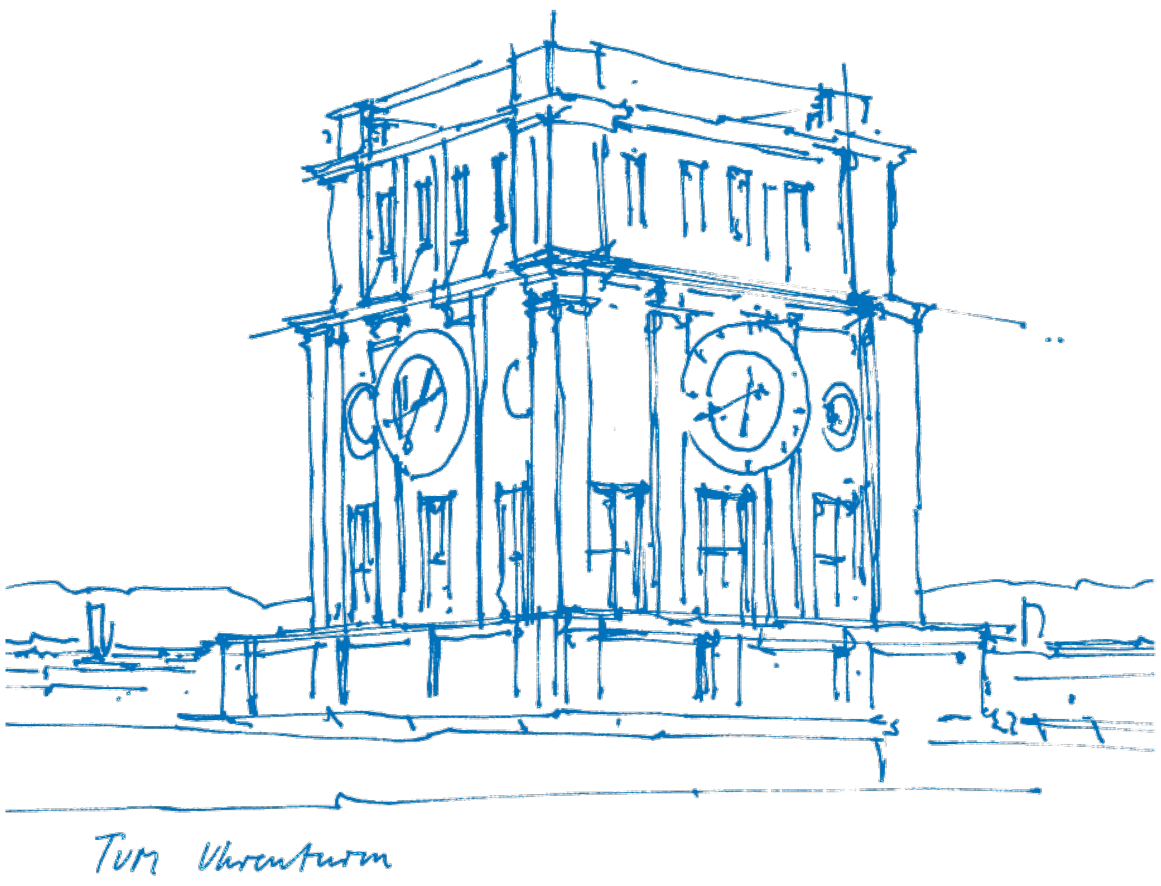


Version 09.05.2022

Camera Models

Image Formation

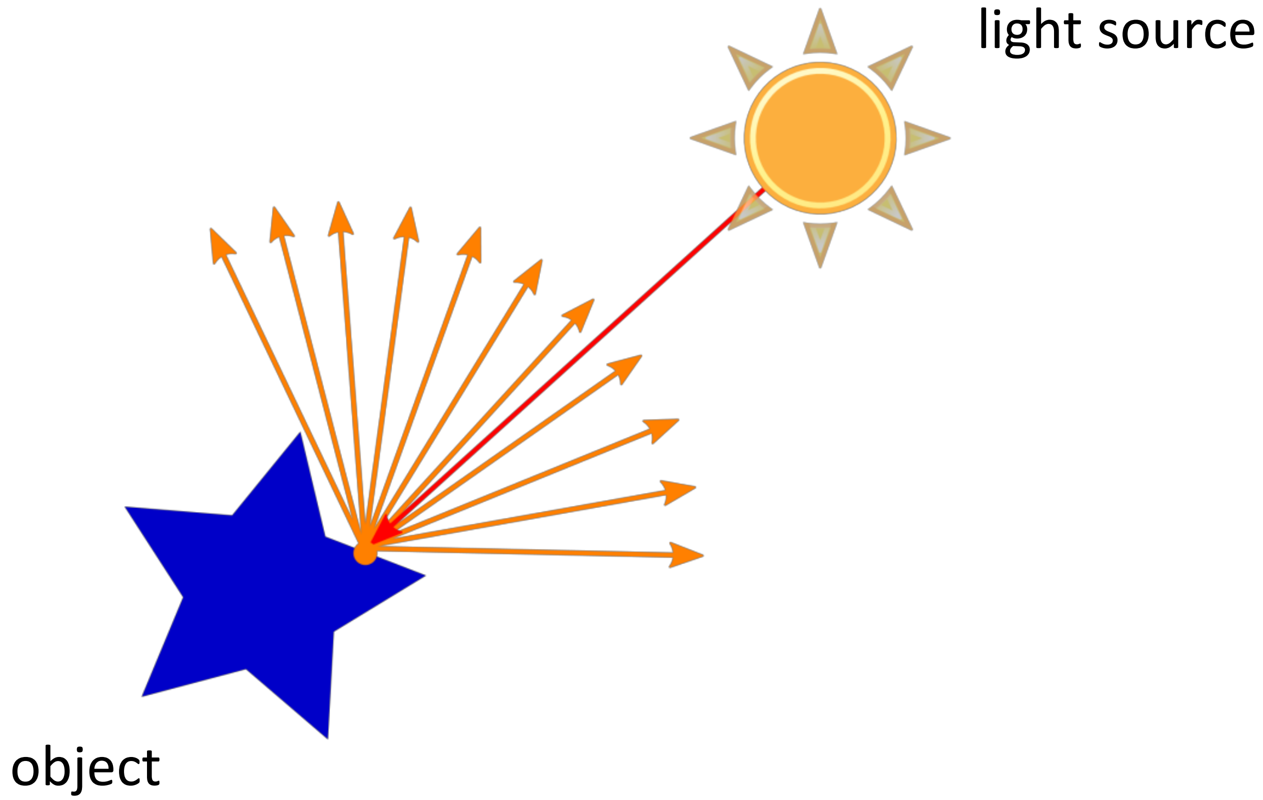


Image Formation

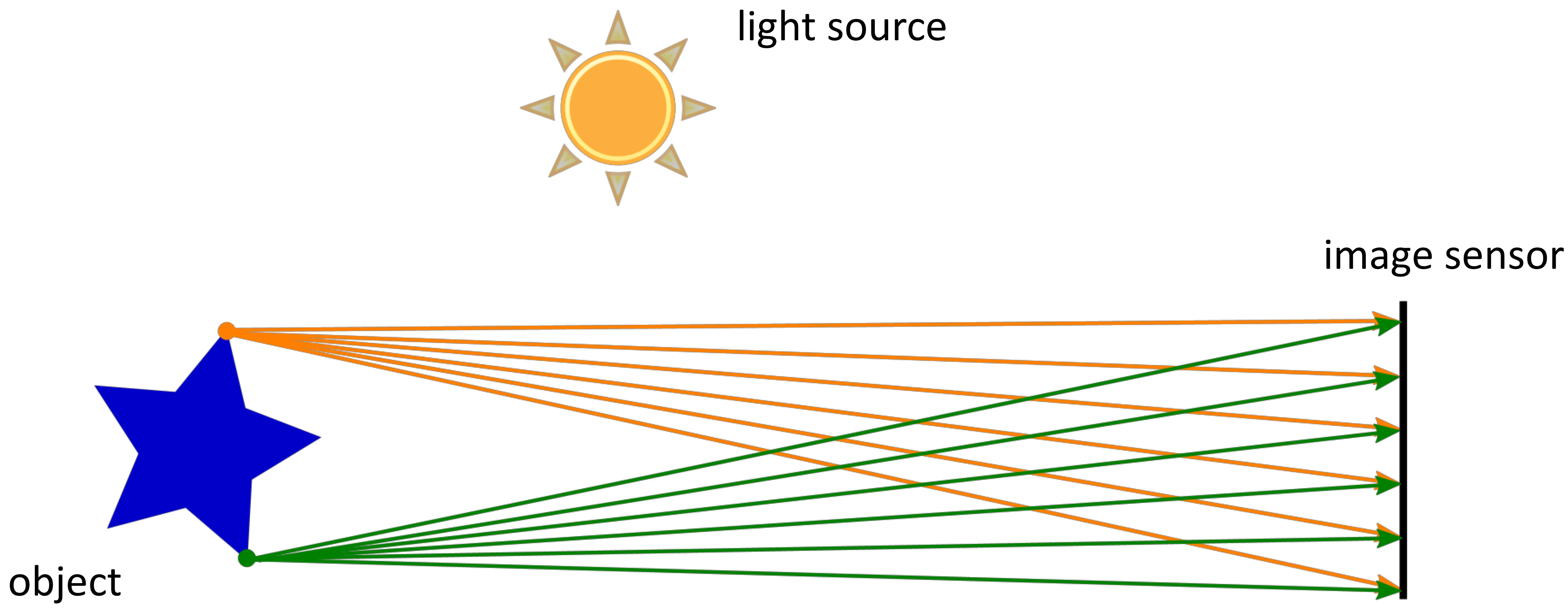
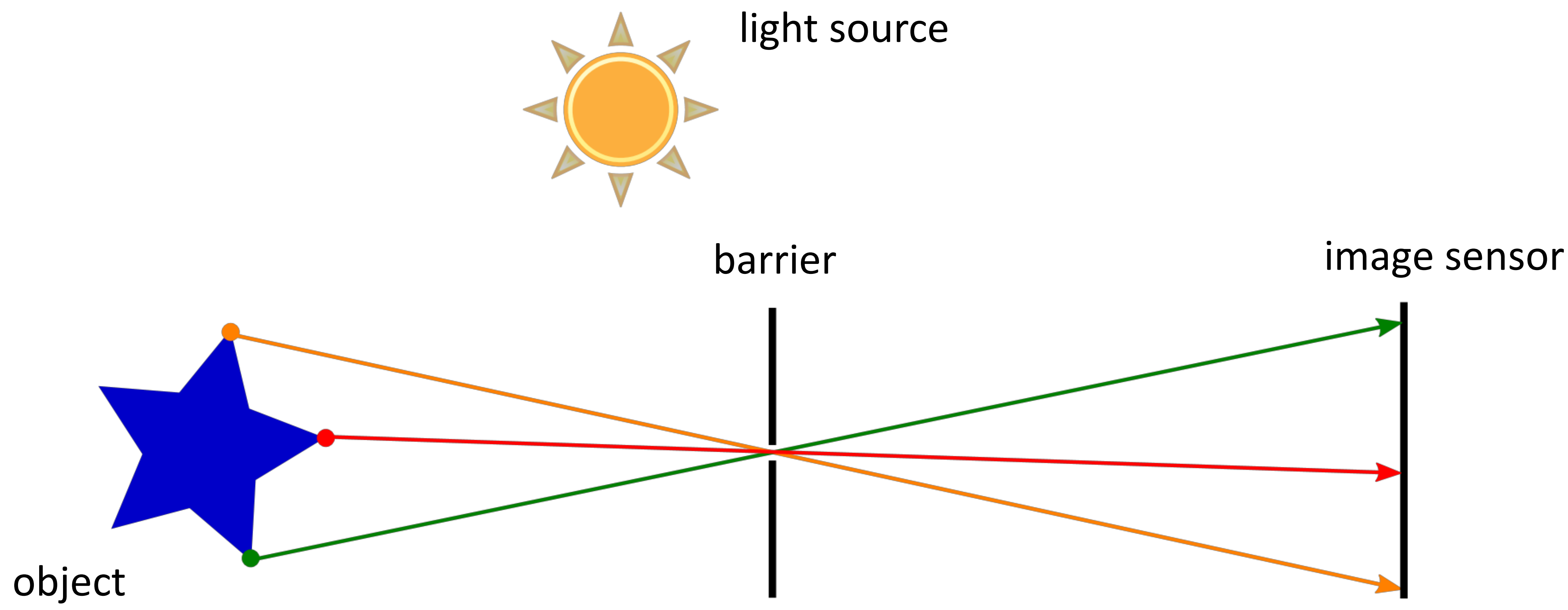
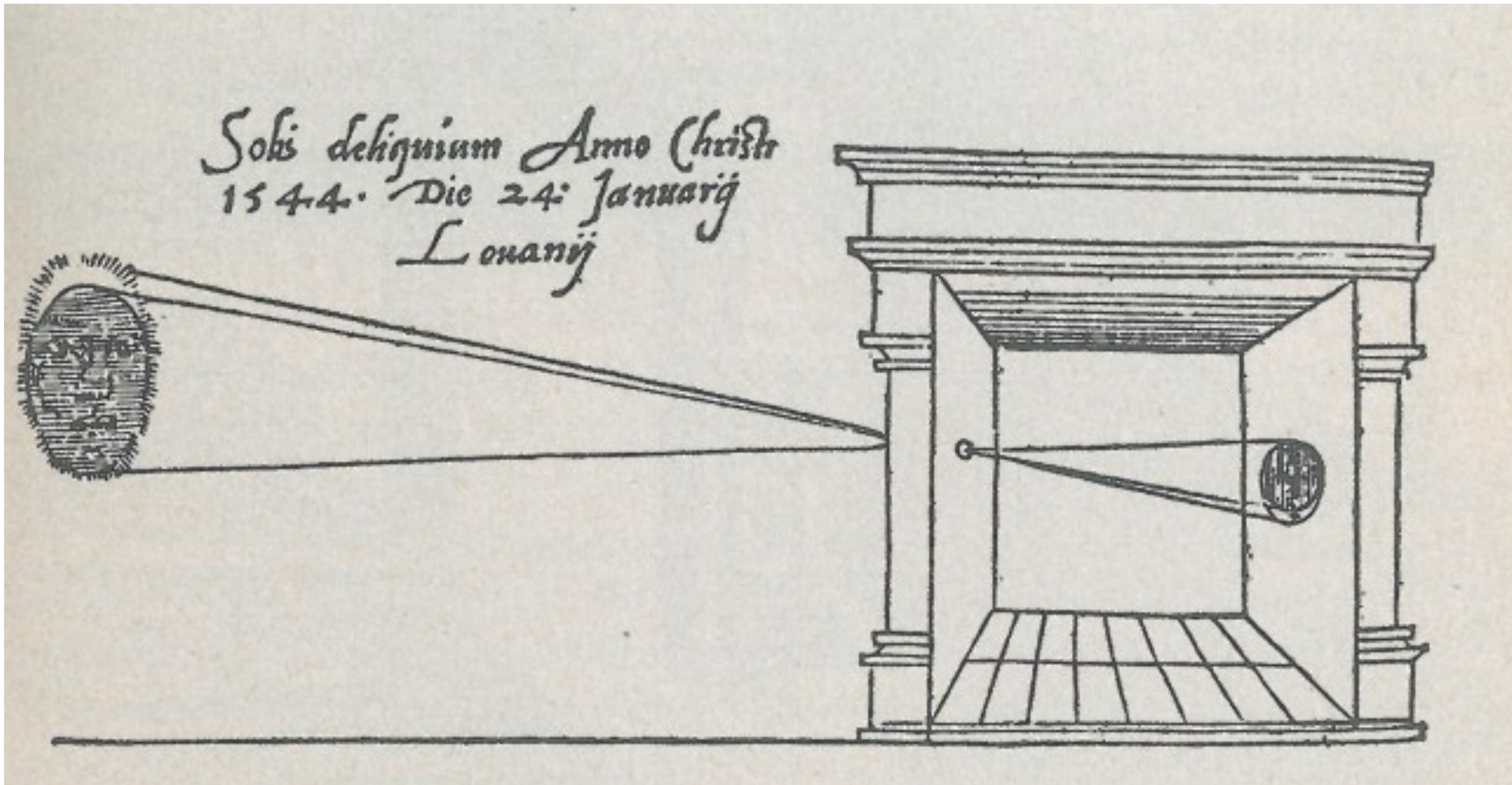


Image Formation



Camera Obscura



First published picture of camera obscura in Gemma Frisius' 1545 book De Radio Astronomica et Geometrica

Pinhole Camera Model

- Camera coordinate frame attached to the center of (0,0) pixel.
  - X - horizontal axis
  - Y - vertical axis downwards
  - Z - forward

Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y\right]^T$$

Projection:

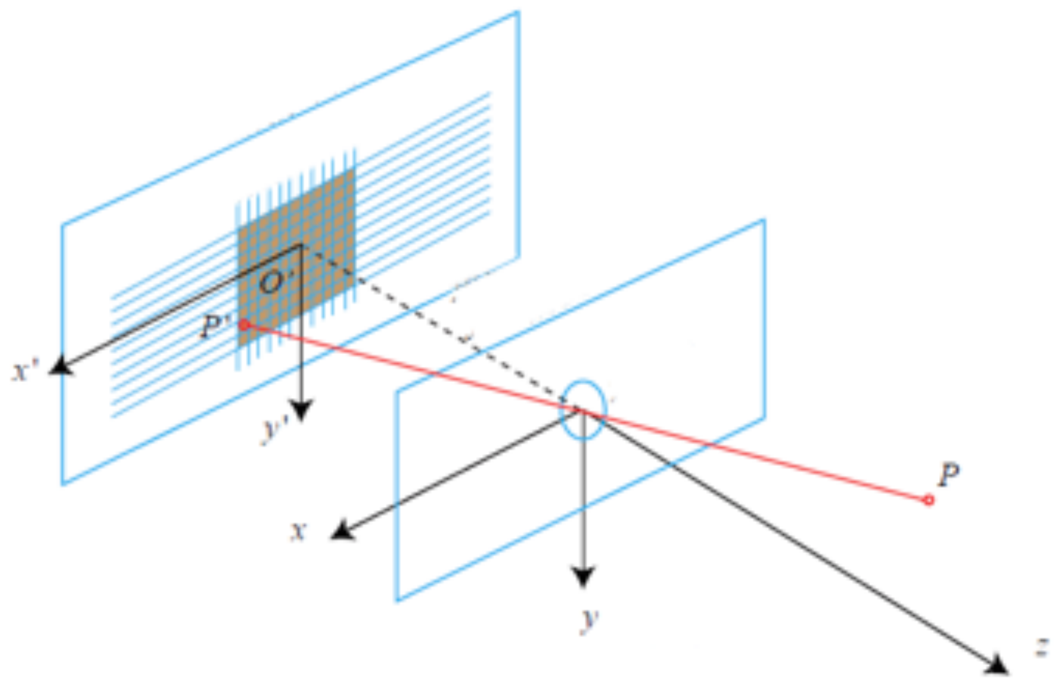
$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \frac{x}{z} \\ f_y \frac{y}{z} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix}$$

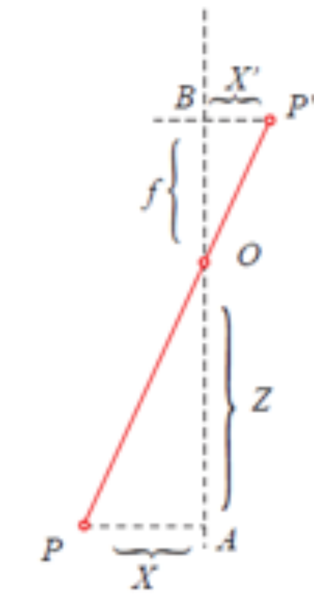
Unprojection:

$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \frac{1}{\sqrt{m_x^2 + m_y^2 + 1}} \begin{bmatrix} m_x \\ m_y \\ 1 \end{bmatrix}$$

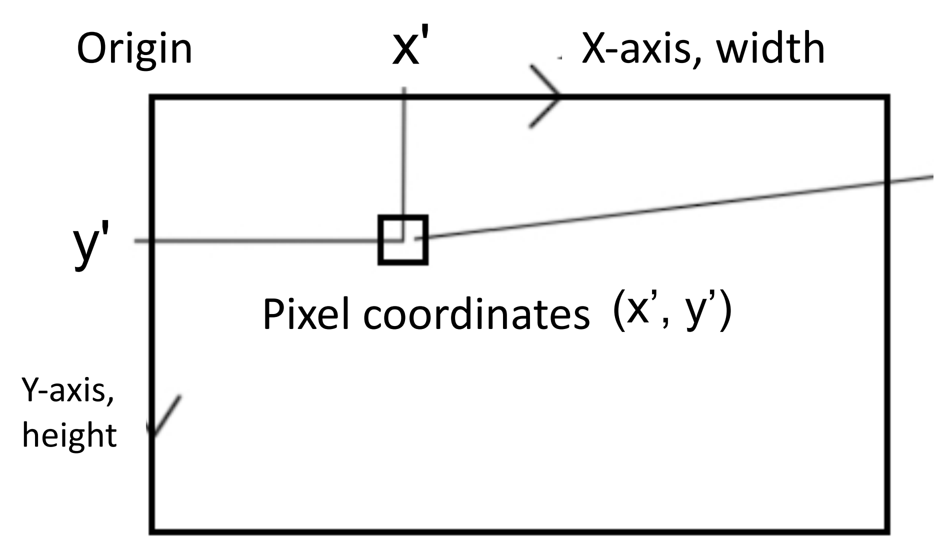
$$m_x = \frac{u - c_x}{f_x},$$

$$m_y = \frac{v - c_y}{f_y}.$$



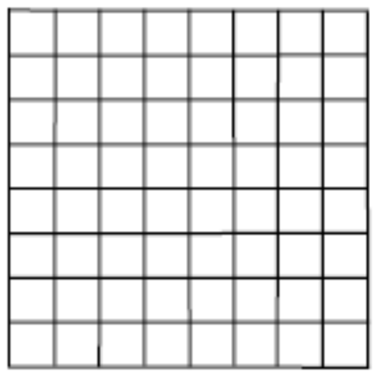


Top View

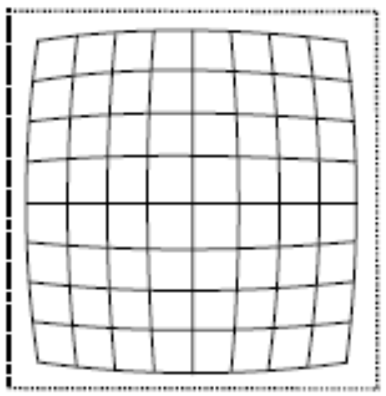


Image

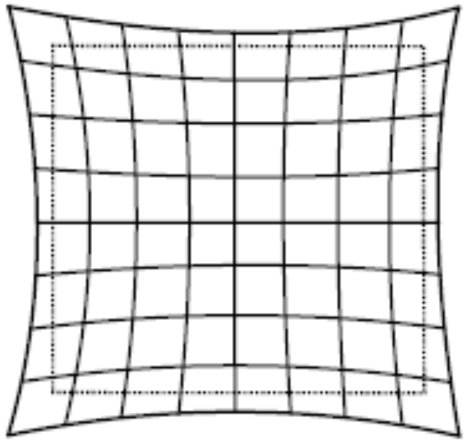
Optical Lens: Distortion and Aperture



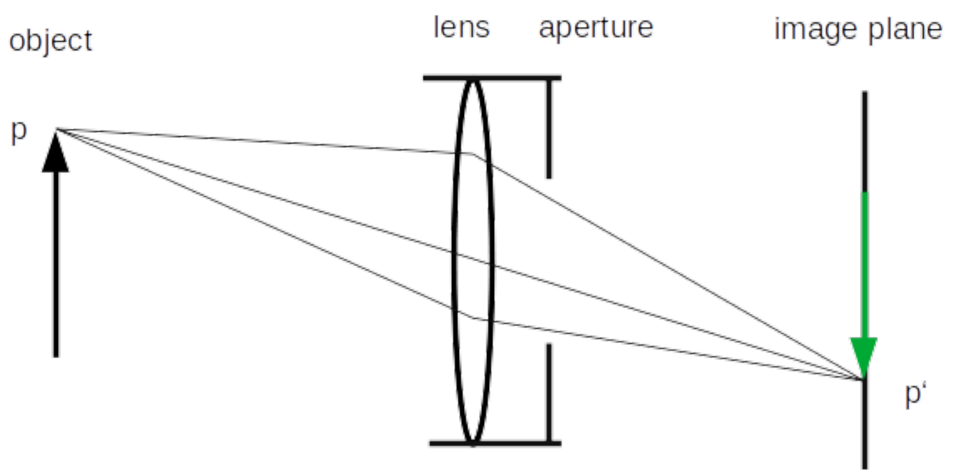
Original image



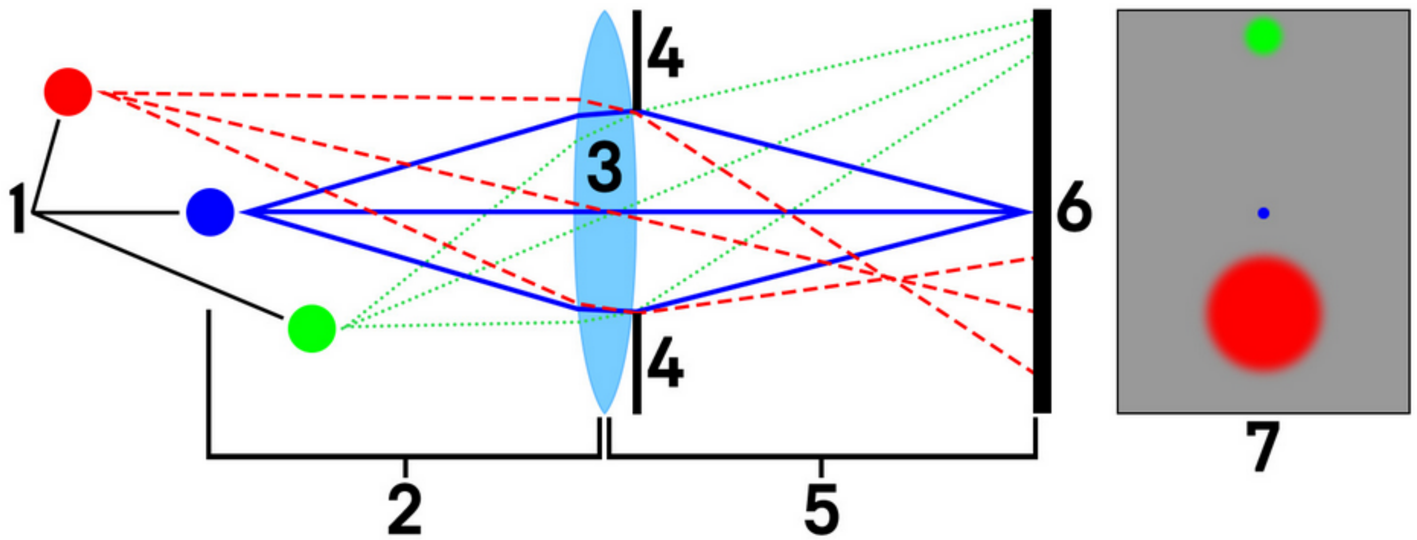
Barrel distortion



Pincushion distortion



Lens Property: All rays from point p (on the object) intersect in p‘ (on the other side of the lens)



Blue: perfect focus, Green: less focused
Red: out of focus
[source: unrealengine 5.0 docs]

Large FOV and Navigation

Z. Zhang, H. Rebecq, C. Forster, D. Scaramuzza “**Benefit of Large Field-of-View Cameras for Visual Odometry**”
IEEE International Conference on Robotics and Automation (ICRA), Stockholm, 2016.

Distortion

Pinhole-Undistorted

- Pinhole
  - Fast projection and unprojection
  - Not suitable for > 180°
  - Bad numeric properties > 120°

Original Image

- More complex model
  - Working with “raw” image
  - No issues with large FOV
  - Possible to optimize intrinsics online

(Extended) Unified Camera Model

Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y, \alpha, \beta\right]^T$$

Projection:

$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \frac{x}{\alpha d + (1-\alpha) z} \\ f_y \frac{y}{\alpha d + (1-\alpha) z} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix},$$

$$d = \sqrt{\beta(x^2 + y^2) + z^2}.$$

Unprojection:

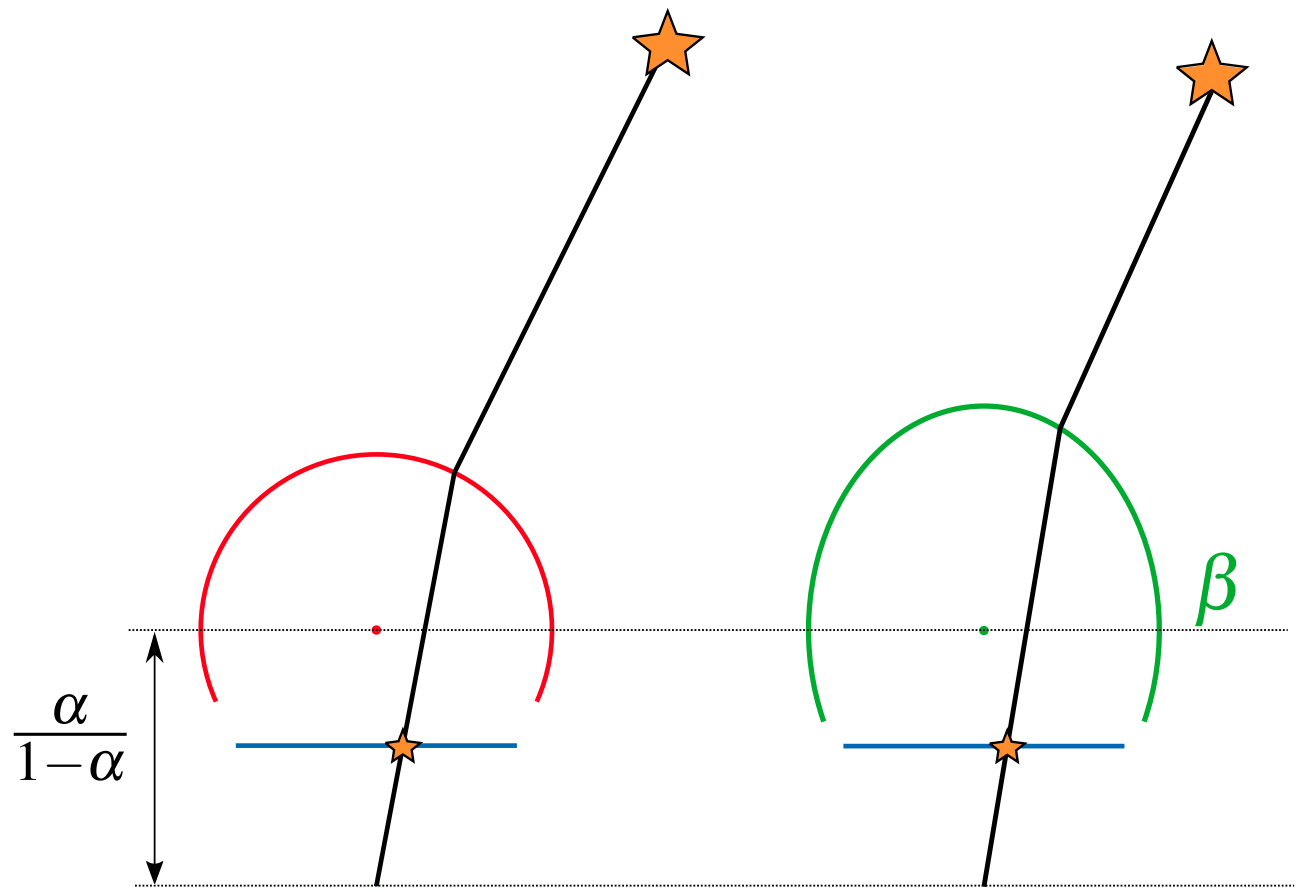
$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \frac{1}{\sqrt{m_x^2 + m_y^2 + m_z^2}} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix},$$

$$m_x = \frac{u - c_x}{f_x},$$

$$m_y = \frac{v - c_y}{f_y},$$

$$r^2 = m_x^2 + m_y^2,$$

$$m_z = \frac{1 - \beta\alpha^2 r^2}{\alpha\sqrt{1 - (2\alpha - 1)\beta r^2} + (1 - \alpha)},$$



Kannala-Brandt Camera Model

Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y, k_1, k_2, k_3, k_4\right]^T$$

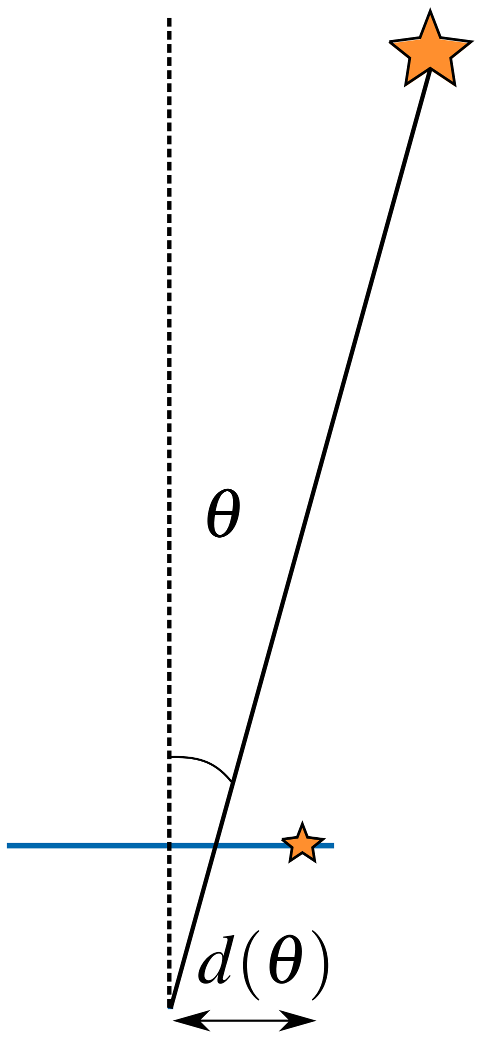
Projection:

$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \; d(\theta) \; \frac{x}{r} \\ f_y \; d(\theta) \; \frac{y}{r} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix},$$

$$r = \sqrt{x^2 + y^2},$$

$$\theta = \text{atan2}(r, z),$$

$$d(\theta) = \theta + k_1\theta^3 + k_2\theta^5 + k_3\theta^7 + k_4\theta^9 \, .$$



Unprojection:

$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \begin{bmatrix} \sin(\theta^*) \; \frac{m_x}{r_u} \\ \sin(\theta^*) \; \frac{m_y}{r_u} \\ \cos(\theta^*) \end{bmatrix},$$

$$m_x = \frac{u - c_x}{f_x},$$

$$m_y = \frac{v - c_y}{f_y},$$

$$r_u = \sqrt{m_x^2 + m_y^2},$$

$$\theta^* = d^{-1}(r_u),$$

Double Sphere Camera Model

Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y, \xi, \alpha\right]^T$$

Projection:

$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \dfrac{x}{\alpha d_2 + (1-\alpha)(\xi d_1 + z)} \\ f_y \dfrac{y}{\alpha d_2 + (1-\alpha)(\xi d_1 + z)} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix},$$

$$d_1 = \sqrt{x^2 + y^2 + z^2},$$

$$d_2 = \sqrt{x^2 + y^2 + (\xi d_1 + z)^2}.$$

Unprojection:

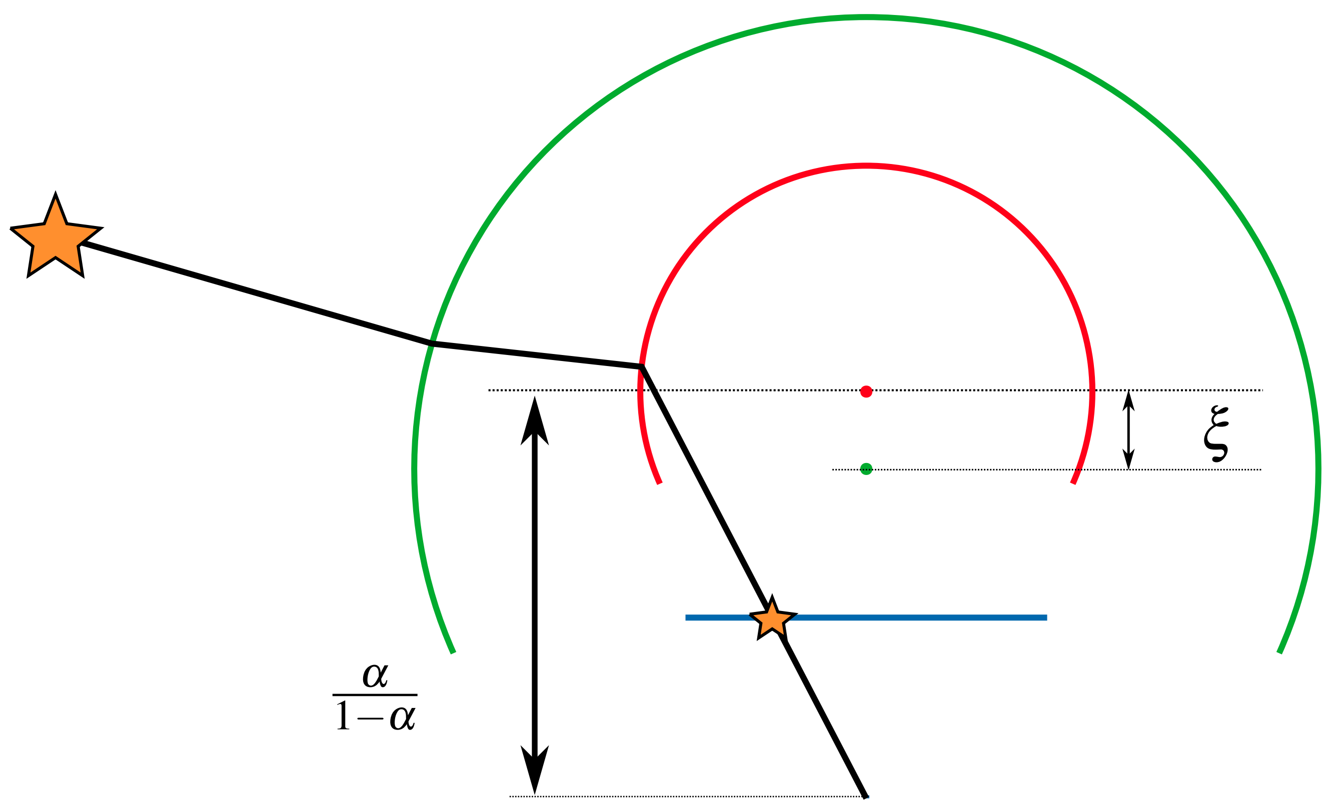
$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \frac{m_z \xi + \sqrt{m_z^2 + (1-\xi^2) r^2}}{m_z^2 + r^2} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \xi \end{bmatrix},$$

$$m_x = \frac{u - c_x}{f_x},$$

$$m_y = \frac{v - c_y}{f_y},$$

$$r^2 = m_x^2 + m_y^2,$$

$$m_z = \frac{1 - \alpha^2 r^2}{\alpha \sqrt{1 - (2\alpha - 1) r^2} + 1 - \alpha}.$$



The Double Sphere Camera Model (V. Usenko, N. Demmel and D. Cremers), *In Proc. of the Int. Conference on 3D Vision (3DV)*, 2018. [arXiv:1807.08957]

Camera Models Code

```
template <typename Scalar>
class PinholeCamera : public AbstractCamera<Scalar> {
 public:
  …
  typedef Eigen::Matrix<Scalar, 2, 1> Vec2;
  typedef Eigen::Matrix<Scalar, 3, 1> Vec3;
  typedef Eigen::Matrix<Scalar, N, 1> VecN;

  PinholeCamera() { param.setZero(); }
  PinholeCamera(const VecN& p) { param = p; }
  …
  virtual Vec2 project(const Vec3& p) const {
    const Scalar& fx = param[0];
    const Scalar& fy = param[1];
    const Scalar& cx = param[2];
    const Scalar& cy = param[3];

    const Scalar& x = p[0];
    const Scalar& y = p[1];
    const Scalar& z = p[2];

    Vec2 res;
    // TODO SHEET 2: implement camera model
    return res;
  }

  virtual Vec3 unproject(const Vec2& p) const {
    const Scalar& fx = param[0];
    const Scalar& fy = param[1];
    const Scalar& cx = param[2];
    const Scalar& cy = param[3];

    Vec3 res;
   // TODO SHEET 2: implement camera model
    return res;
  }
  …
  EIGEN_MAKE_ALIGNED_OPERATOR_NEW
 private:
  VecN param;
};
```

- Avoid using std::pow() function to maintain the precision. For example, if you need to compute x^2 use multiplication: $x * x$.
- If your compiler complains about Jet types try changing the constants in projection and unprojection functions to Scalar(<constant>). For example, Scalar(1) instead of 1.
- You can use Newton's method for finding roots to compute a root of the polynomial given a good initialization. Usually 3-5 iterations should be enough for the optimization to converge.
- You can use Horner's method to efficiently compute polynomials.

Optimization

Maximum a Posteriori Estimation

Given a set of parameters $x = \{x_1, ... x_n\}$ and a set of observations that depend on the parameters $z = \{z_1, ... z_m\}$ we want estimate the value of x that is most likely to result in these observations:

$$x^* = \underset{x}{\text{argmax}}\ P(x \mid z),$$

This estimate of the parameters x^* is called the Maximum a posteriori (MAP) estimation.

We can rewrite the probability using the Bayes' Rule:

$$\underbrace{P(x \mid z)}_{\text{Posteriori}} = \frac{\overbrace{P(z \mid x)}^{\text{Likelihood}}\overbrace{P(x)}^{\text{Prior}}}{P(z)}.$$

We can drop the denominator, because it does not depend on x.

$$x^* = \underset{x}{\text{argmax}}\ P(z \mid x)P(x).$$

"Which state it is most likely to produce such measurements?"

From MAP to Least Squares

- From MAP to least squares problem
- If we assume that the measurements are independent the joint PDF can be factorized:

$$P(z\,|\,x) = \prod_{k=0}^{K} P(z_k\,|\,x)$$

- Let's consider a single observation: $z_k = h(x) + v_k,$
  - Affected by Gaussian noise: $v_k \sim N(0,\ Q_k)$

- The observation model gives us a conditional PDF:

$$P(z_k\,|\,x) = N(h(x),\ Q_k)$$

- How do we estimate x ?

From MAP to Least Squares

- Gaussian Distribution (matrix form)

$$P(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right).$$

- Take negative logarithm from both sides:

$$-\ln P(x) = \frac{1}{2}\ln((2\pi)^p|\Sigma|) + \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu).$$

- Maximum of $P(x)$ is equivalent to the minimum of $-\ln P(x)$.

From MAP to Least Squares

- Batch least squares
- Formulate residual function:

$$r_k = z_k - h(x).$$

- Maximizing of $P(x)$ is equivalent to the minimizing the sum of squared residuals:

$$E(x) = \frac{1}{2} \sum_k r_k^T Q r_k.$$

From MAP to Least Squares

- Some notes:
  - Because of noise, when we take the estimated trajectory and map into the models, they won't fit perfectly
  - Then we adjust our estimation to get a better estimation (minimize the error)
  - The error distribution is affected by noise distribution (information matrix)
- Structure of the least square problem
  - Sum of many squared errors
  - The dimension of total state variable may be high
  - But single error item is easy (only related to two states in our case)
  - If we use Lie group and Lie algebra, then it's a non-constrained least square

$$E(x) = \frac{1}{2} \sum_k r_k^T Q r_k$$

Least Squares

- How to solve a least square problem?
  - Non-linear, discrete time, non-constrained

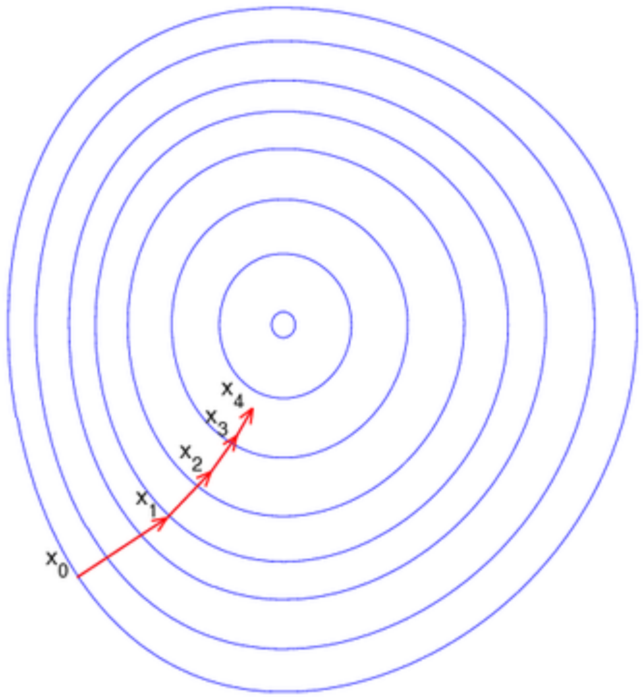
- Let's start from a simple example
  - Consider minimizing a squared error:
  - When $E(x)$ is simple, just solve:

$$E(x) = \frac{1}{2}\sum_k r_k(x)^T r_k(x) = \frac{1}{2} r(x)^T r(x)$$

$$\frac{\partial E(x)}{\partial x} = 0$$

- And we will find the maxima/minima/saddle points

Least Squares

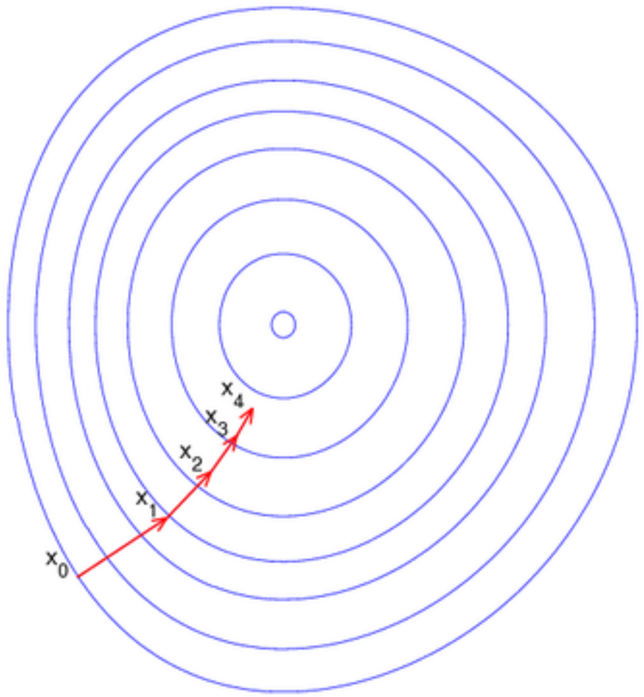


- When $E(x)$ is a complicated function:
  - $\dfrac{\partial E(x)}{\partial x} = 0$ is hard to solve
  - We use iterative methods

- Iterative methods
  1. Start from an initial estimate x_0
  2. At iteration n, we find an increment Δx_n that minimizes $E(x_n + \Delta x_n)$.
  3. If the change in error function is small enough, stop (converged).
  4. If not, set $x_{n+1} = x_n + \Delta x_n$ and iterate to step 2.

Gradient Descent

- How to find the increment?
- First order methods - Gradient Descent
  - Taylor expansion of the objective function
  - $E(x + \Delta x) = E(x) + G(x)\Delta x$

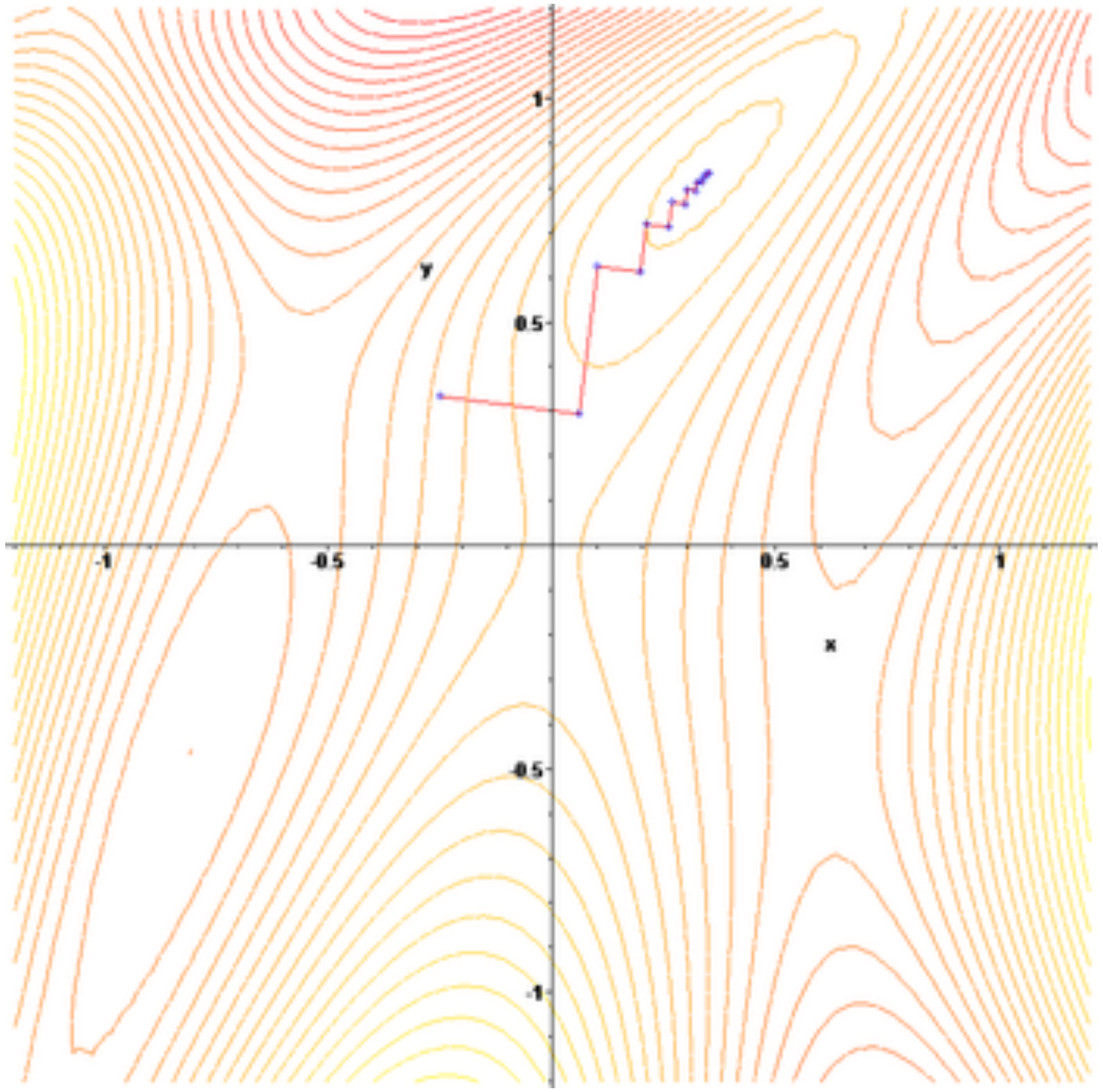


The update step:

$\Delta x = -\alpha G(x)$

Gradient Descent Performance

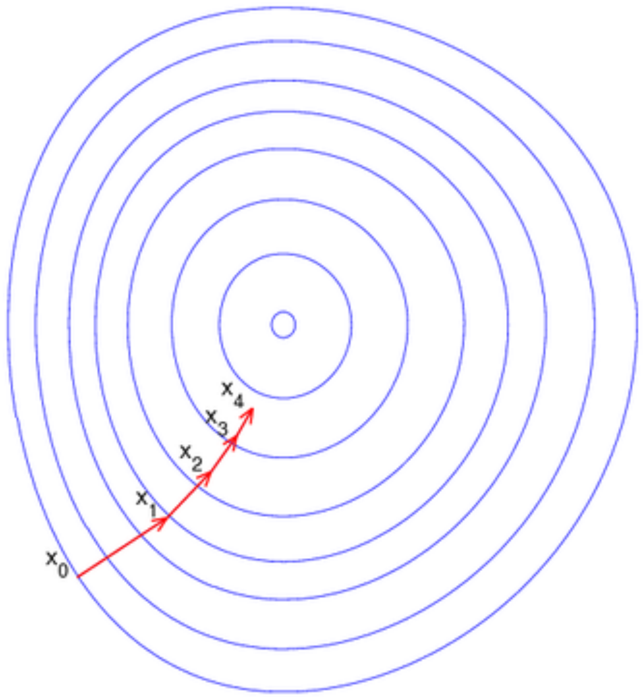
Zig-zag in steepest descent:



- Other shortcomings:
  - Slow convergence speed
  - Even slower when close to minimum

Second Order Methods

- Second order methods
  - Taylor expansion of the objective function
  - $E(x + \Delta x) = E(x) + G(x)\Delta x + \Delta x^T H(x)\Delta x$
  - Setting $\dfrac{\partial E(x + \Delta x)}{\partial \Delta x} = 0$

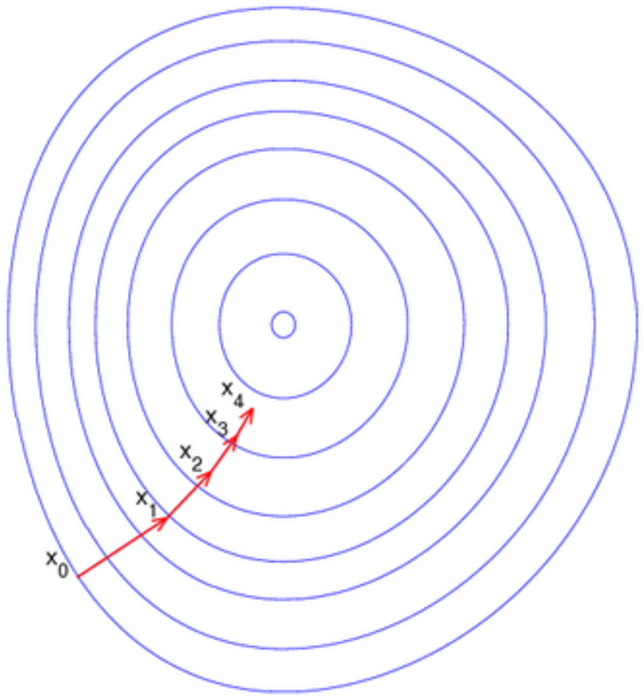


The update step:

$$H(x)\Delta x = -\,G(x) \implies \Delta x = -\,H^{-1}(x)\;G(x)$$

This is called Newton's method.

Second Order Methods for Least Squares



- Second order method converges more quickly than first order methods
- But the Hessian matrix may be hard to compute

- Can we avoid the Hessian matrix and also keep second order's convergence speed? Yes, for least squares problems, there exists faster options:
  - Gauss-Newton
  - Levenberg-Marquardt

Gauss-Newton Method

- Gauss-Newton
  - Taylor expansion of $r(x)$: $r(x + \Delta x) \simeq r(x) + J(x)\Delta x$
  - Then the squared error becomes:

$$E(x + \Delta x) = \frac{1}{2} r(x)^T r(x) + \Delta x^T J(x)^T r(x) + \frac{1}{2} \Delta x^T J(x)^T J(x) \Delta x$$

$$= F(x) + \Delta x^T J(x)^T r(x) + \frac{1}{2} \Delta x^T J(x)^T J(x) \Delta x$$

If we set $\dfrac{\partial E(x + \Delta x)}{\partial \Delta x} = 0$ we get:

$$J^T(x) J(x) \Delta x = -J(x)^T r(x) \implies \Delta x = -\left(J^T(x) J(x)\right)^{-1} J(x)^T r(x)$$

$\simeq H(x)$ Newton's Method $\simeq G(x)$

Gauss-Newton Method

- Gauss-Newton uses $J^T(x)J(x)$ as an approximation of the Hessian
  - Avoids the computation of $H(x)$ in the Newton's method

- But $J^T(x)J(x)$ is only semi-positive definite
- $H(x)$ maybe singular when $J^T(x)J(x)$ has null space

Levenberg-Marquardt Method

- Trust region approach: approximation is only valid in a region
- Evaluate if the approximation is good:

$$\rho = \frac{r(x + \Delta x) - r(x)}{J(x)\Delta x}.$$

Real descent/approx. descent

- If ρ is large, increase the region
- If ρ is small, decrease the region

- LM optimization:

$$E(x + \Delta x) = \frac{1}{2} r(x + \Delta x)^T r(x + \Delta x) + \lambda \|\Delta x\|^2$$

  - Assume the approximation is only good within a region
  - λ controls the region based on ρ

Levenberg-Marquardt Method

- Trust region problem:

$$E(x + \Delta x) = \frac{1}{2} r(x + \Delta x)^T r(x + \Delta x) + \lambda \|\Delta x\|^2$$

- Expand it just like in GN case, the incremental is:

$$\Delta x = -\left(J^T(x) J(x) + \lambda I\right)^{-1} J(x)^T r(x)$$

- The λI part makes sure that Hessian is positive definite.
- When $\lambda = 0$ LM becomes GN.
- When $\lambda \to \infty$ LM becomes gradient descent.

Other Methods

- Dog-leg method
- Conjugate gradient method
- Quasi-Newton’s method
- Pseudo-Newton’s method
- …

- You can find more in optimization books if you are interested

- In SLAM/SfM/VO, Gauss-Netwton and Levenberg-Marquardt are used to solve camera motion, optical-flow, etc.

More details:

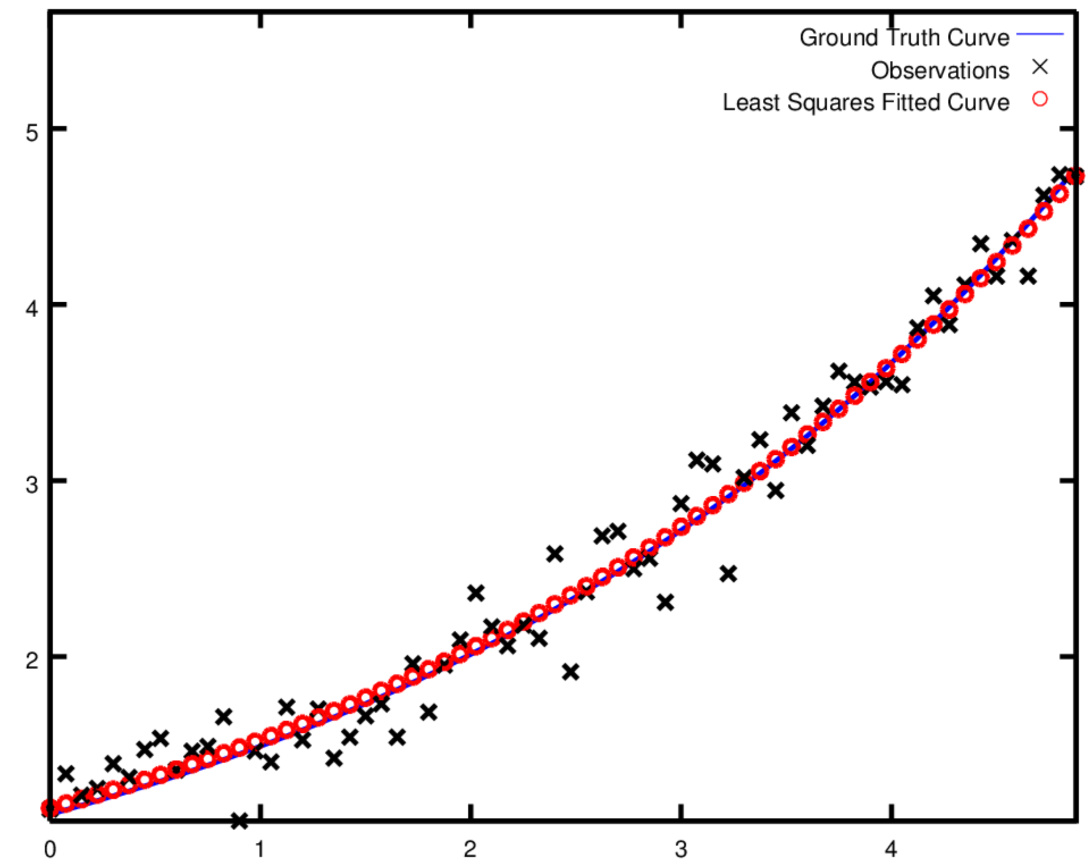
Triggs B, McLauchlan PF, Hartley RI, Fitzgibbon AW. Bundle adjustment—a modern synthesis. InInternational workshop on vision algorithms 1999 Sep 20 (pp. 298-372). Springer, Berlin, Heidelberg.

Ceres

- We will use Ceres for least-squares optimization.

- Tutorial: http://ceres-solver.org/tutorial.html

- Curve fitting example: $y = \exp(mx + c)$
- Observations: a set of (x, y) pairs
- Parameters to estimate: m, c.



Ceres

- Define your residual class as a functor (overload the () operator)

```
struct ExponentialResidual {
  ExponentialResidual(double x, double y)
      : x_(x), y_(y) {}

  template <typename T>
  bool operator()(const T* const m, const T* const c, T*
residual) const {
    residual[0] = T(y_) - exp(m[0] * T(x_) + c[0]);
    return true;
  }

 private:
  // Observations for a sample.
  const double x_;
  const double y_;
};
```

Ceres

- Build the optimization problem

```
double m = 0.0;
double c = 0.0;

Problem problem;
for (int i = 0; i < kNumObservations; ++i) {
  CostFunction* cost_function =
       new AutoDiffCostFunction<ExponentialResidual, 1, 1, 1>(
           new ExponentialResidual(data[2 * i], data[2 * i + 1]));
  problem.AddResidualBlock(cost_function, NULL, &m, &c);
}
```

- With auto-diff, Ceres will compute the Jacobians for you

Ceres

- Finally solve it by calling the Solve() function and get the result summary
- You can set some parameters like number of iterations, stop conditions or the linear solver type.

```
// Run the solver!
 Solver::Options options;
 options.linear_solver_type = ceres::DENSE_QR;
 options.minimizer_progress_to_stdout = true;
 Solver::Summary summary;
 Solve(options, &problem, &summary);

 std::cout << summary.BriefReport() << "\n";
 std::cout << "m : " << m
           << “c : " << c << "\n";
```

Least-Squares Summary

- In the maximum a posteriori estimation we estimate all the state variable given using a set of noisy measurements.
- The MAP estimation problem with Gaussian noise can be reformulated into a least square problem
- It can be solved by iterative methods: Gradient Descent, Newton's method, Gauss-Newton or Levernberg-Marquardt.

Exercise 2

- We want to estimate
  - Poses of the camera setup with respect to pattern
  - Intrinsic parameters of both cameras
  - Extrinsic parameters (rigid body transformation from one camera to the other)

- Minimizing the projection residuals:

$$r = u_j - \pi(R_{cw} p_w^j + t_{cw}, \mathbf{i}),$$

- u_j - detection of the corner j in the image.
- p_w^j - 3D coordinates in the world (pattern) coordinate frame
- $\mathbf{i}$ - intrinsic parameter of the camera
- R_{cw}, t_{cw} - rigid body transformation from the world (pattern) coordinate frame to the camera coordinate frame.
- π - is the projection function

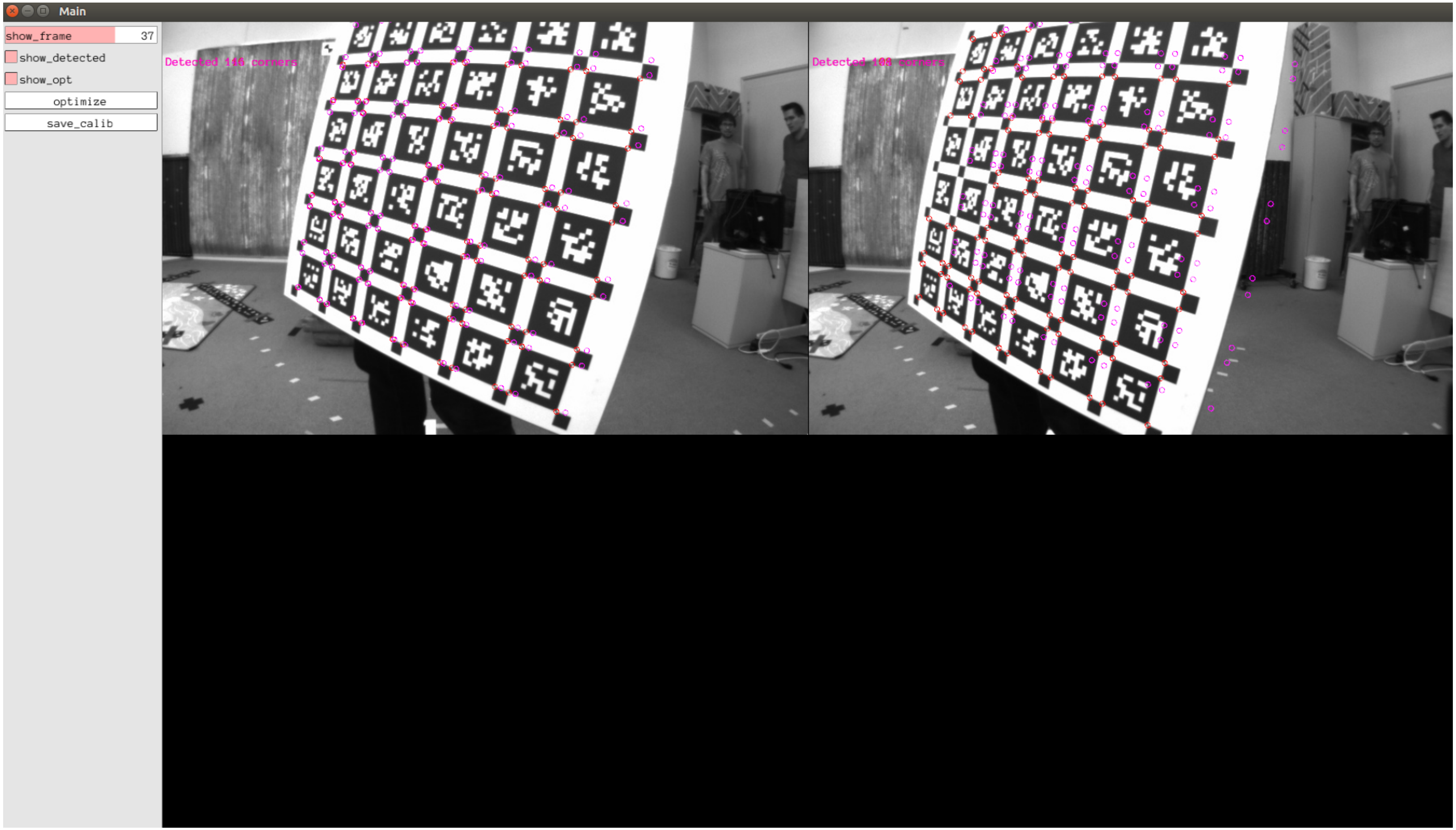
- Corner points are detected using Apriltags

E. Olson. AprilTag: A robust and flexible visual fiducial system. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA)*, pages 3400–3407. IEEE, May 2011.

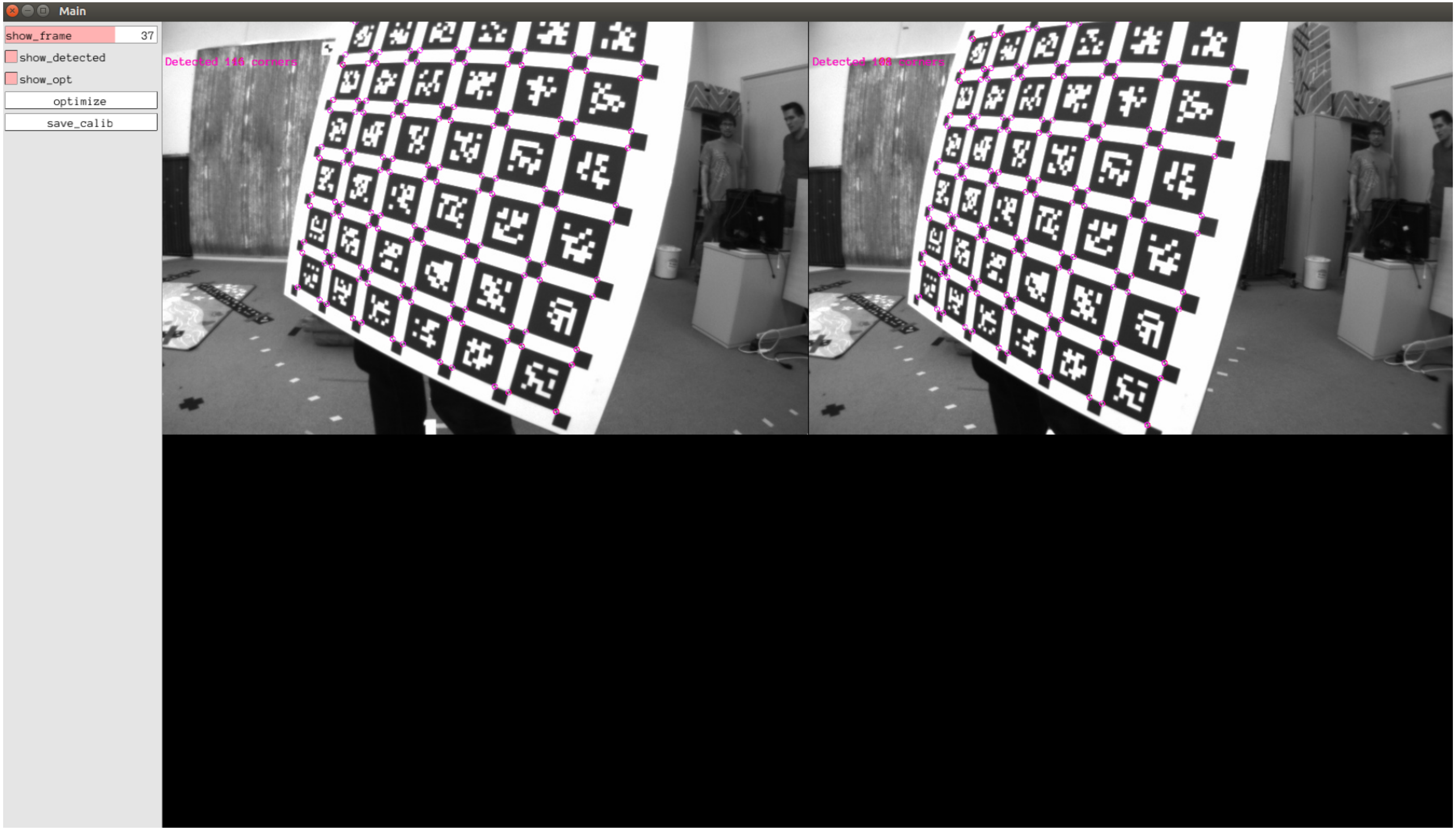
Exercise 2 Residual

```
struct ReprojectionCostFunctor {
  EIGEN_MAKE_ALIGNED_OPERATOR_NEW
  ReprojectionCostFunctor(const Eigen::Vector2d& p_2d,
                          const Eigen::Vector3d& p_3d,
                          const std::string& cam_model)
      : p_2d(p_2d), p_3d(p_3d), cam_model(cam_model) {}

  template <class T>
  bool operator()(T const* const sT_w_i, T const* const sT_i_c,
                  T const* const sIntr, T* sResiduals) const {
    Eigen::Map<Sophus::SE3<T> const> const T_w_i(sT_w_i);
    Eigen::Map<Sophus::SE3<T> const> const T_i_c(sT_i_c);

    Eigen::Map<Eigen::Matrix<T, 2, 1>> residuals(sResiduals);
    const std::shared_ptr<AbstractCamera<T>> cam =
        AbstractCamera<T>::from_data(cam_model, sIntr);

    // TODO SHEET 2: implement the rest of the functor
    return true;
  }

  Eigen::Vector2d p_2d;
  Eigen::Vector3d p_3d;
  std::string cam_model;
};
```

Exercise 2


Main
show_frame
37
show_detected
show_opt
optimize
save_calib

Exercise 2


Main
show_frame 37
show_detected
show_opt
optimize
save_calib
Detected 116 corners
Detected 108 corners

Exercise 2

- Use camera models presented here to get initial projections
- Implement the projection function
- Implement the residual.
- Set up optimization problem. Use local parametrization where necessary.

- Test different models. How well do they fit the lens?