Computer Vision II: Multiple View Geometry (IN2228)

Chapter 02 Motion and Scene Representation
(Part 2 Lie Group and Lie Algebra)

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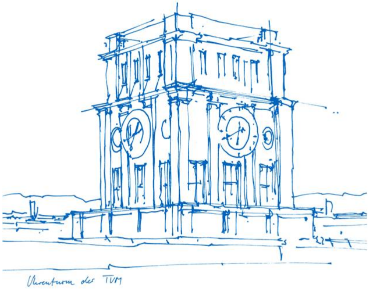
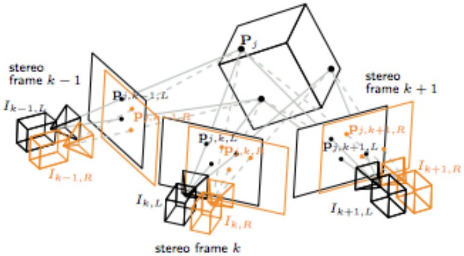


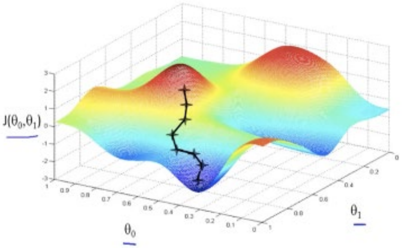
Today's Outline

- Motivation
- Concepts of Group
- Lie Group and Lie Algebra

Motivation

- ✓ Optimize the initial estimation (expression->computation->optimization)
- ✓ Find a constraint-free optimization strategy





$$\mathrm{SO}(n) = \{\mathbf{R} \in \mathbb{R}^{n\times n} | \mathbf{R}\mathbf{R}^T = \mathbf{I}, \det(\mathbf{R}) = 1\}$$

Orthogonality constraint

Concepts of Group

- Definition and properties of group

A group is an algebraic structure of **one set** plus **one operator**.

$$G = (A, \cdot)$$

"•" denotes the operator instead of multiplication

A group should satisfy the following conditions (e.g., **integer set** plus **addition**)

- Closure: $\forall a_1, a_2 \in A,\ a_1 \cdot a_2 \in A.$
- Associative law: $\forall a_1, a_2, a_3 \in A,\ (a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3).$
- Identity element: $\exists a_0 \in A,\ \text{s.t.}\ \forall a \in A,\ a_0 \cdot a = a \cdot a_0 = a.$
- Inverse: $\forall a \in A,\ \exists a^{-1} \in A,\ st\ a \cdot a^{-1} = a_0.$

"0" for addition
"1" for multiplication

x_0 and $-x_0$ for addition
x_0 and $1/x_0$ for multiplication

Concepts of Group

- Common groups

✓ General Linear group GL(n). The invertible n*n matrix with matrix multiplication.

✓ Special Orthogonal Group SO(n) or the rotation matrix group, where SO(2) and SO(3) is the most common.

- **Rotation matrix set** plus **matrix multiplication** form a group.
- Unit element: Identity matrix
- Identity element: $R * R^{-1} = I$

✓ Special Euclidean group SE(n) described earlier, such as SE(2) and SE(3).

Lie Group and Lie Algebra

➢ Lie Group

- Lie Group refers to a group with continuous (smooth) properties.
- SO(n) and SE(n) are continuous in real space since we can intuitively imagine that a rigid body moving continuously in the space, so they are all Lie Groups.
- Two matrices in SO(3) or SE(3) can be multiplied, but not added, which affects the derivate computation.

$$\tilde{\mathbf{b}} = \mathbf{T}_1 \tilde{\mathbf{a}}, \ \tilde{\mathbf{c}} = \mathbf{T}_2 \tilde{\mathbf{b}} \quad \Rightarrow \tilde{\mathbf{c}} = \mathbf{T}_2 \mathbf{T}_1 \tilde{\mathbf{a}}$$

Multiplication on SE(3)

Lie Group and Lie Algebra

- Introduction to Lie Algebra (not very formal, just for understanding)

R(t) denotes a rotation of a camera that changes continuously over time

$$\mathbf{R}(t)\mathbf{R}(t)^T = \mathbf{I}.$$

By taking derivatives with respect to the time t, we obtain

$$\dot{\mathbf{R}}(t)\mathbf{R}(t)^T + \mathbf{R}(t)\dot{\mathbf{R}}(t)^T = 0.$$

$\dot{\mathbf{R}}$ represents the derivative

We move the second term to the right side and rewrite it based on the transpose

$$\dot{\mathbf{R}}(t)\mathbf{R}(t)^T = -\left(\dot{\mathbf{R}}(t)\mathbf{R}(t)^T\right)^T.$$

skew-symmetric matrix

Lie Group and Lie Algebra

➢ Introduction to Lie Algebra

$$\mathbf{a} = (a_1 \; a_2 \; a_3)^{\mathsf{T}} \quad \mathbf{a}^{\wedge} = \mathbf{A} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}, \quad \mathbf{A}^{\vee} = \mathbf{a}.$$

$$\left[[x_1, x_2, x_3]^{\top}\right]_{\times} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

For writing simplification, we denote the skew-symmetric matrix by

$$\dot{\mathbf{R}}(t)\mathbf{R}(t)^{T} = \phi(t)^{\wedge}.$$

3*1 vector

What's the meaning of φ(t)^?

Right multiply both sides by R(*t*), we have

$$\dot{\mathbf{R}}(t) = \phi(t)^{\wedge}\mathbf{R}(t)$$

We use the first-order Taylor series around t_0 to expand R(t)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\mathbf{R}(t) \approx \mathbf{R}(t_0) + \dot{\mathbf{R}}(t_0)(t - t_0) = \mathbf{I} + \phi(t_0)^{\wedge}(t).$$

$$t_0 = 0$$

$$\mathbf{R}(0) = \mathbf{I}$$

Lie Group and Lie Algebra

- Introduction to Lie Algebra

0->x0

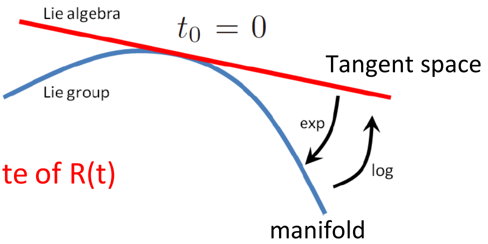
$\mathbf{R}(t)$ (t=0+t)

$= \mathbf{I} + \phi(t_0)^\wedge (t).$

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

t->Δx

Definition of derivative



φ(t)^ corresponds to the derivate of R(t)

Lie Group and Lie Algebra

➢ Definition of Lie Algebra

Lie Algebra so(3)

$$\mathfrak{so}(3) = \left\{ \phi \in \mathbb{R}^3 \text{ or } \boldsymbol{\Phi} = \phi^\wedge \in \mathbb{R}^{3\times 3} \right\}. \qquad \boldsymbol{\Phi} = \phi^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

Its relationship to SO(3) is given by the exponential map:

$$\mathbf{R} = \exp(\phi^\wedge).$$

Detailed formula will be introduced later

Through it, we map any vector in so(3) to a rotation matrix in SO(3).

Lie Group and Lie Algebra

- Definition of Lie Algebra

Lie Algebra se(3)

$$\mathfrak{se}(3) = \left\{ \xi = \begin{bmatrix} \rho \\ \phi \end{bmatrix} \in \mathbb{R}^6, \rho \in \mathbb{R}^3, \phi \in \mathfrak{so}(3), \xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ \mathbf{0}^T & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \right\}.$$

- ✓ The first three dimensions are "translation part" $\boldsymbol{\rho}$ (but keep in mind that the meaning is different from the translation in the matrix).
- ✓ The second part is a rotation part φ, which is essentially the so(3) element.

Lie Group and Lie Algebra

➢ Definition of Lie Algebra

How to calculate exp(φ∧), i.e., an exponential map of a matrix?

$$\exp\left(\phi^{\wedge}\right) = \exp\left(\theta \mathbf{n}^{\wedge}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\theta \mathbf{n}^{\wedge}\right)^{n}$$

3D vector with the norm θ and unit direction n.

...

$$\exp(\theta \mathbf{n}^{\wedge}) = \cos\theta \mathbf{I} + (1 - \cos\theta)\mathbf{n}\mathbf{n}^{T} + \sin\theta \mathbf{n}^{\wedge}.$$

This shows that so(3) is actually the rotation vector, and the exponential map is just Rodrigues' formula.

Lie Group and Lie Algebra

- Definition of Lie Algebra

Conversely, if we define a logarithmic map, we can also map the elements in SO(3) to so(3):

$$\phi = \ln(\mathbf{R})^{\vee} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (\mathbf{R} - \mathbf{I})^{n+1} \right)^{\vee}.$$

Use the properties of the trace to solve the rotation angle and the rotation axis separately

$$\theta = \arccos\left(\frac{\mathrm{tr}(\mathbf{R}) - 1}{2} \right).$$

$$\mathbf{R}\mathbf{n} = \mathbf{n}.$$

The axis n is the eigenvector corresponding to the matrix R's eigenvalue 1.

Lie Group and Lie Algebra

$$\mathbf{R} = \cos\theta \mathbf{I} + (1-\cos\theta)\mathbf{n}\mathbf{n}^T + \sin\theta \mathbf{n}^\wedge$$

Rodrigues' rotation formula

➢ Definition of Lie Algebra

The exponential map on se(3) is described below

$$\exp(\boldsymbol{\xi}^\wedge) = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{1}{n!}(\boldsymbol{\phi}^\wedge)^n & \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\boldsymbol{\phi}^\wedge)^n \boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$\triangleq \begin{bmatrix} \mathbf{R} & \mathbf{J}\rho \\ \mathbf{0}^T & 1 \end{bmatrix} = \mathbf{T}.$$

$$\mathbf{J} = \frac{\sin\theta}{\theta}\mathbf{I} + \left(1 - \frac{\sin\theta}{\theta}\right)\mathbf{a}\mathbf{a}^T + \frac{1-\cos\theta}{\theta}\mathbf{a}^\wedge$$

$$\begin{cases} y_1 = f_1(x_1, \ldots, x_n) \\ \vdots \\ y_n = f_n(x_1, \ldots, x_n), \end{cases}$$

$$\mathbf{J}(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}.$$

Jacobian matrix

- ✓ This formula is similar to the Rodrigues formula but not exactly the same.
- ✓ After passing the exponential map, the translation part is multiplied by a linear Jacobian matrix **J**.

Lie Group and Lie Algebra

- Definition of Lie Algebra

3D Rotation

Lie Group

$SO(3)$

$R \in \mathbb{R}^{3\times3}$

$RR^T = I$

$\det(R) = 1$

Exponential: $\exp(\theta a^\wedge) = \cos\theta I + (1-\cos\theta) aa^T + \sin\theta a^\wedge$

Logarithmic: $\theta = \arccos\frac{tr(R)-1}{2}$ $\quad Ra = a$

Lie Algebra

$\mathfrak{so}(3)$

$\phi \in \mathbb{R}^3$

$$\phi^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix}$$

3D Transform

Lie Group

$SE(3)$

$T \in \mathbb{R}^{4\times4}$

$$T = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$$

Exponential:

$$\exp(\xi^\wedge) = \begin{bmatrix} \exp(\phi^\wedge) & J\rho \\ 0^T & 1 \end{bmatrix}$$

$$J = \frac{\sin\theta}{\theta} I + \left(1 - \frac{\sin\theta}{\theta}\right) aa^T + \frac{1-\cos\theta}{\theta} a^\wedge$$

Logarithmic: $\theta = \arccos\frac{tr(R)-1}{2}$ $\quad Ra = a \quad t = J\rho$

Lie Algebra

$\mathfrak{se}(3)$

$$\xi = \begin{bmatrix} \rho \\ \phi \end{bmatrix} \in \mathbb{R}^6$$

$$\xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix}$$

By courtesy of Dr. Xiang Gao (former member of our group)

Lie Group and Lie Algebra

➢ BCH Formula and Its Approximation

✓ Motivation

BCH formula is the basis of computing derivatives on so(3)

✓ Recap

$$\mathbf{R}_1 + \mathbf{R}_2 \notin SO(3) \qquad \phi_1 + \phi_2 \in \mathfrak{so}(3)$$

✓ Does the addition of two vectors in so(3) correspond to the product of the two matrices on SO(3)? In other words, does the following equation hold?

$$\exp\left(\phi_1^\wedge\right)\exp\left(\phi_2^\wedge\right) = \exp\left(\left(\phi_1 + \phi_2\right)^\wedge\right) \qquad ??$$

More generally, $\ln\left(\exp\left(\mathbf{A}\right)\exp\left(\mathbf{B}\right)\right) = \mathbf{A} + \mathbf{B}$

Lie Group and Lie Algebra

➢ BCH Formula and its Approximation

✓ The above formula does not hold for the matrices. Is there an approximation?

✓ The complete form of the product is given by the Baker-Campbell-Hausdorff formula (BCH formula)

$$\ln(\exp(\mathbf{A})\exp(\mathbf{B})) = \mathbf{A} + \mathbf{B} \quad \times$$

$$\Downarrow$$

$$\ln(\exp(\mathbf{A})\exp(\mathbf{B})) = \mathbf{A} + \mathbf{B} + \overbrace{\underbrace{\frac{1}{2}[\mathbf{A},\mathbf{B}]}_{\text{Lie bracket}} + \frac{1}{12}[\mathbf{A},[\mathbf{A},\mathbf{B}]] - \frac{1}{12}[\mathbf{B},[\mathbf{A},\mathbf{B}]] + \cdots}^{\text{small terms}} \quad \checkmark$$

Lie Group and Lie Algebra

➢ BCH Formula and its Approximation

✓ BCH formula can be used to tackle $\exp(\phi_1^\wedge)\exp(\phi_2^\wedge)$

Small perturbation

✓ In practice, small items can be ignored when taking derivatives. At this time, BCH has a linear approximation

Small variable on so(3)

$$\ln\left(\exp(\phi_1^\wedge)\exp(\phi_2^\wedge)\right)^\vee \approx \begin{cases} \mathbf{J}_l(\phi_2)^{-1}\phi_1 + \phi_2 & \text{when } \phi_1 \text{ is a small amount,} \checkmark \\ \mathbf{J}_r(\phi_1)^{-1}\phi_2 + \phi_1 & \text{when } \phi_2 \text{ is a small amount.} \end{cases}$$

Small rotation on SO(3)

$$\mathbf{J}_l = \mathbf{J} = \frac{\sin\theta}{\theta}\mathbf{I} + \left(1 - \frac{\sin\theta}{\theta}\right)\mathbf{a}\mathbf{a}^T + \frac{1-\cos\theta}{\theta}\mathbf{a}^\wedge$$

Jacobian matrix introduced before

Lie Group and Lie Algebra

$$\ln\left(\exp\left(\phi_1^\wedge\right)\exp\left(\phi_2^\wedge\right)\right)^\vee \approx \begin{cases} \mathbf{J}_l(\phi_2)^{-1}\phi_1 + \phi_2 & \text{when } \phi_1 \text{ is a small amount,} \\ \mathbf{J}_r(\phi_1)^{-1}\phi_2 + \phi_1 & \text{when } \phi_2 \text{ is a small amount.} \end{cases}$$

➢ BCH Formula and its Approximation

✓ Suppose we have a rotation R. Its corresponding Lie algebra is φ.

✓ We assign R a small perturbation ΔR. Its Corresponding Lie algebra is Δφ.

✓ On Lie group, the perturbation result is $\Delta\mathbf{R}\cdot\mathbf{R}$. On the Lie algebra, according to the BCH approximation, we have $\mathbf{J}_l^{-1}(\phi)\Delta\phi + \phi$

By combining them, we have

$$\exp\left(\Delta\phi^\wedge\right)\exp\left(\phi^\wedge\right) = \exp\left(\left(\phi + \mathbf{J}_l^{-1}\left(\phi\right)\Delta\phi\right)^\wedge\right)$$

Summary

- Motivation
- Concepts of Group
- Lie Group and Lie Algebra

Thank you for your listening!
If you have any questions, please come to me :-)