Announcements before Class

- Supplementary Exercise Codes

- For some knowledge (e.g., computing normalized image coordinates and computing the relative camera pose from the absolute camera pose), our exercise sessions do not involve them due to time and space limit.

- As compensation, we are preparing some MATLAB codes to help you review some important knowledge that is not covered by our exercise sessions but is still highlighted in our review document (for Chapters 01--05).

- Please note that it is optional for you to check these codes.

- For details, please refer to https://www.moodle.tum.de/mod/forum/discuss.php?d=431822
Announcements before Class

- Update about Exercise Session Schedule

- For 2D-2D Geometry, we originally intend to introduce basic configuration (two views) and a more complex configuration (multiple views).
- Due to time limit, we skip the case of **multiple views** (last year, Prof. Cremers also skipped this part).
- Our lecture schedule remains unchanged. However, **we cancel the Exercise 7**.

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**Core part**

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**Exercises**

- Wed 14.06.2023  Exercise 6: Reconstruction from two views
- Wed 21.06.2023  Exercise 7: Reconstruction from **multiple views**
- Wed 05.07.2023  Exercise 8: Direct Image Alignment
- Wed 12.07.2023  Exercise 9: Direct Image Alignment
Announcements before Class

➢ Notations and Formulas

✓ Problems
There are some inconsistent symbols in slides (e.g., intrinsic parameters).
Some equations and formulas are in the image format.

✓ Reasons
Some equations are copied from different academic papers.

✓ Solutions
We are adjusting notations and formulas. Due to limited time, the progress is relatively slow. If some of them affect your understanding, please let me know (via email or Moodle) and I will prioritize their update.
Announcements before Class

- Correction for Rank Analysis (Tsai’s Method)
  
  ✓ $Q$ $(2n \times 12)$ should have rank 11 to have a unique (up-to-scale) non-zero solution of vector $M$. If rank equals 12, we only have zero solution.
  
  ✓ Dimension of null space is 1. Vector $M$ can be expressed by a single basis vector multiplied by an arbitrary scalar.

\[ Q \cdot M = 0 \]

- Please refer to the updated page 14/48 of Chapter 04.
Explanation for Tsai’s Method

- **QR Decomposition**

- Assume that we have computed M matrix based on DLT. We aim to recover intrinsic matrix K, rotation matrix R, and translation vector t.

- First step: re-write the right-hand side based on distributive law

\[
M = K \begin{bmatrix} R & t \end{bmatrix} = \begin{bmatrix} KR & Kt \end{bmatrix}
\]

- Second step: use RQ decomposition to decompose the known \(KR\) to obtain K and R. Note: QR decomposition is a generic term that may refer to QR, QL, RQ, and LQ decompositions, with L being a lower triangular matrix. (https://en.wikipedia.org/wiki/QR_decomposition#Relation_to_RQ_decomposition)

- Third step: compute t based on the known \(Kt\) and the estimated K.
Today’s Outline

- Five-point Method
- Pose Recovery from Essential matrix
- 1D Correspondence Search
Five-point Method

- Review on Rank and Null Space

An example

Computing a 3D direction $L$ orthogonal to normal $N$

$$\langle N, L\rangle = 0$$

$$N^TL = 0$$

$\{b_1, b_2\}$

Num(unknowns) = Rank($N$) + Dim(null space)

i.e., dimension of $L$

$3 \quad 1 \quad 2$

Dot product A linear system Basis of null space

$N$: 1*3
Five-point Method

- Two Properties of Essential Matrix

- Recap on definition of essential matrix

\[ E = [t]^\wedge R \]

- Properties of essential matrix

\[ det(E) = 0 \]

\[ EE^T E - \frac{1}{2} \text{trace}(EE^T) E = 0 \]  

Scalar \[ \text{Scalar} \]

Scalar \[ \text{Scalar} \]

\[ det([t]^\wedge) = 0 \quad \text{Rank}([t]^\wedge) = 2 \]

\[ det(A_1 A_2 \cdots A_n) = det(A_1) \cdot det(A_2) \cdots det(A_n) \]

3*3 zero matrix

Properties will not be asked in the exam
Five-point Method

Revisiting Eight-point Method

✓ Linear System

\[
\begin{bmatrix}
u_1^1 u_1^1 & u_1^2 v_1^1 & u_2^1 v_1^1 & v_1^2 u_1^1 & v_1^1 v_1^1 & v_2^1 u_1^1 & v_1^1 u_1^1 & v_1^1 v_1^1 & 1 \\
u_2^1 u_1^2 & u_2^2 v_1^2 & u_2^2 u_1^2 & u_2^2 v_1^2 & v_2^2 v_1^2 & v_2^2 u_1^2 & v_2^2 u_1^2 & v_2^2 v_1^2 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
u_1^n u_1^n & u_2^n v_1^n & u_2^n u_1^n & u_2^n v_1^n & v_2^n v_1^n & v_2^n u_1^n & v_2^n u_1^n & v_2^n v_1^n & 1
\end{bmatrix}
\begin{bmatrix}e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8\end{bmatrix} = 0
\]

\[Q^9 e = 0 \text{ \ N*9}\]

• Each correspondence can provide one constraint.
• The minimal case of Q is 8*9 if we neglect the inherent constraints of elements of e.
• The minimal case is 5 correspondences if we consider these constraints. (we no longer solve a linear system)

✓ Definition of E matrix

\[E = [T_x]R \quad \text{essential matrix}\]

• R has three degrees of freedom
• T has two degrees of freedom
• E has five degrees of freedom
• Rank of Q equals five

\[\text{Dim}(e) = \text{Rank}(Q) + \text{Dim(null space)}\]

\[
\begin{align*}
5 & \quad 9 \\
9 & \quad 5 \\
5 & \quad 4
\end{align*}
\]
Five-point Method

- Polynomial Generation

- Basis of null space
  \[ X, Y, Z, W \quad \text{Dim(null space)} = 4 \]
  Known 9D basis vectors computed based on the known 5*9 coefficient matrix

- Linear expression of vector \( e \)
  \[ e = xX + yY + zZ + wW \]
  \( x, y, z \) are unknown coefficients \( w = 1 \)

- Constraints of \( E \) matrix
  \[ \det(E) = 0 \]
  \[ EE^T E - \frac{1}{2} \text{trace}(EE^T) E = 0 \]
  1 constraint \( 9 \) constraints (only two of them are linear independent)

\[ Qe = 0 \]
Five-point Method

- Polynomial Generation

- Polynomial System

**Unknown** vector with respect to x, y, z

\[ A p = 0 \]

- Known coefficient matrix

\[ \begin{bmatrix} x^3, y^3, x^2y, xy^2, x^2z, x^2, y^2z, y^2, xz, x, y, yz, x, xz, xy, xz, x, yz, y, x, z, z, z, 1 \end{bmatrix} . \]

A 10-th degree univariate polynomial with respect to z

\[ e = xX + yY + zZ + wW \]

W = 1

Computed coefficients

Solving this non-linear system will not be asked in the exam
Pose Recovery from Essential matrix

- Essential Matrix Decomposition

✓ Assume that we have obtained an essential matrix $E$. Recall that essential matrix $E$ encodes the camera pose information. We aim to recover rotation and translation from the matrix $E$.

$E = [T_x]R$ \textit{essential matrix}

✓ Recap on singular value decomposition

$T = U \Sigma V^{-1}$

Composed of orthogonal basis vectors
Pose Recovery from Essential matrix

- Essential Matrix Decomposition

- Lemma: singular value of 3*3 Essential matrix satisfies the form $[\sigma, \sigma, 0]^T$

We do not provide proof here. If you have interest, you can check the reference [1]

- Decomposition of essential matrix E

Mathematical and geometric forms

$$E = U \Sigma V^T = U \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T = [t]^R.$$  

Geometric form

Mathematical form

Link: [https://www.robots.ox.ac.uk/~vgg/hzbook/](https://www.robots.ox.ac.uk/~vgg/hzbook/)
Pose Recovery from Essential matrix

- Essential Matrix Decomposition

✓ Can we directly extract R and t from SVD result?

\[ E = U \Sigma V^T = U \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T = [t]^R \]

We cannot directly extract t and R (no skew-symmetric matrix)

We have to further transform it

Skew-symmetric matrix

✓ Rewrite \( \Sigma \) by matrix multiplication (not unique)

\[ \Sigma = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
Pose Recovery from Essential matrix

- Essential Matrix Decomposition

✓ Decomposition of essential matrix

\[
E = U \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T = U \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sum V^T
\]

Skew-symmetric matrix

Introducing an identity matrix for derivation

Associative law

[[t] \wedge R]
Pose Recovery from Essential matrix

- Essential Matrix Decomposition

✓ Rotation and translation results

\[
\begin{bmatrix} t_1 \end{bmatrix}^\wedge = U \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T = [U \begin{bmatrix} 0 \\ 0 \\ -a \end{bmatrix}]^\wedge
\]

Translation

\[
\begin{bmatrix} t_1 \end{bmatrix} = U \begin{bmatrix} 0 \\ 0 \\ -a \end{bmatrix} = \begin{bmatrix} U_0 & U_1 & U_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -a \end{bmatrix} = U_2 a
\]

Rotation

\[
R_1 = U \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^T
\]

Translation
Pose Recovery from Essential matrix

- Essential Matrix Decomposition

✓ Another decomposition of $\Sigma$ leads to another result

$$\Sigma = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} 
  t_2 = U \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} = U_2 a = -t_1 \\
  R_2 = U \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} V^T \end{cases}$$

Another rotation and translation results
Pose Recovery from Essential matrix

Essential Matrix Decomposition

✓ A more concise expression of rotation and translation

Rotation derived before

\[
R_1 = U R_z \left( \frac{\pi}{2} \right) V^T, \quad R_2 = U R_z \left( -\frac{\pi}{2} \right) V^T
\]

\[
\Sigma = \begin{bmatrix}
0 & 0 & 0 \\
0 & a & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Translation derived before

\[
[t_1]^\wedge = U \begin{bmatrix}
0 & a & 0 \\
-a & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} U^T
\]
Pose Recovery from Essential matrix

- Essential Matrix Decomposition

Four possible solutions: There exists only one solution where points are in front of both cameras.

Different translations

Different rotations

Different rotations and translations

3D reconstruction will be introduced in the next class.
Pose Recovery from Essential matrix

Application of Fundamental Matrix

Can \( R, T, K_1, K_2 \) be extracted from \( F \)?

- In general we cannot achieve this since infinite solutions exist.
- However, if the coordinates of the **principal points** of each camera are known and the two cameras have the same **focal length** \( f \) in pixels, then \( R, T, f \) can be determined uniquely.
- This is an advanced knowledge. We will not introduce it in our class.
Pose Recovery from Essential matrix

➢ Application of Fundamental Matrix

✓ If we do not use Fundamental matrix to recover camera pose, what is its application?
  • We do not need to normalize image points.
  • We can use consensus constraint w.r.t. fundamental matrix to remove outliers.
1D Correspondence Search

- Overview

- Reviewing drawback of brute-force matching
- 1D search based on epipolar constraint (application of epipolar lines)
1D Correspondence Search

➢ Review and Motivation

Given a point, \( p_L \), in the left image, how do we find its correspondence, \( p_R \), in the right image? A straightforward strategy is brute-force matching (search) strategy.
1D Correspondence Search

➢ Review and Motivation

Brute-force Matching: compare each candidate patch from the image with all possible candidate patches from the right image.
1D Correspondence Search

- Review and Motivation

This 2D exhaustive search is computationally very expensive!
1D Correspondence Search

Problem Formulation

Can we make the correspondence search 1D?

- The epipolar line is the projection of a back projected ray $\pi^{-1}(p)$ onto the other camera image.
- Potential matches for $p$ have to lie on the corresponding epipolar line $l'$.

\[ \pi^{-1}(p) = \lambda K^{-1} p \]
1D Correspondence Search

Problem Formulation

- Corresponding points must lie along the epipolar lines: this constraint is called **epipolar constraint**.

- The epipolar constraint reduces correspondence problem to 1D search along the epipolar line.
1D Correspondence Search

Problem Formulation

Thanks to the epipolar constraint, corresponding points can be searched for along epipolar lines. Accordingly, the computational cost reduced to 1 dimension.
1D Correspondence Search

Example Configurations of Epipolar Lines

Fundamental: All the epipolar lines intersect at the epipole

$\pi^{-1}(p) = \lambda K^{-1} p$

Epipolar plane

$e = \text{epipoles}$

$C_i, C_r$

Basis of epipolar geometry

Intersection

Rotation axis, i.e., baseline
1D Correspondence Search

- Example 1: Converging Cameras

✓ A classical case (non-parallel principal axes)
As the position of the 3D point $P$ changes, the epipolar lines rotate about the baseline
1D Correspondence Search

Example 2: Identical and Horizontally-Aligned Cameras

✓ A special case (principal axes are parallel, and orthogonal to moving direction/baseline)
Parallel epipolar lines do not intersect (or, the intersection lies at the infinity).
1D Correspondence Search

Example 3: Forward Motion

Another special case (principal axes are aligned to moving direction)
Epipolar lines radiating from the epipole (coordinates remain unchanged)
Summary

- Five-point Method
- Pose Recovery from Essential Matrix
- 1D Correspondence Search
Thank you for your listening!
If you have any questions, please come to me :-)