1. Write down the matrices $M \in SE(3) \subset \mathbb{R}^{4 \times 4}$ representing the following transformations:

   (a) Translation by the vector $T \in \mathbb{R}^3$.
   
   (b) Rotation by the rotation matrix $R \in \mathbb{R}^{3 \times 3}$.
   
   (c) Rotation by $R$ followed by the translation $T$.
   
   (d) Translation by $T$ followed by the rotation $R$.

2. Let $M_1, M_2 \in \mathbb{R}^{3 \times 3}$. Please prove the following:

   $$ x^\top M_1 x = x^\top M_2 x \quad \text{iff} \quad M_1 - M_2 \text{ is skew-symmetric} $$

   for all $x \in \mathbb{R}^3$ (i.e. $M_1 - M_2 \in so(3)$)

   *Info:* The group $SO(3)$ is called a **Lie group**. The space $so(3) = \{ \hat{\omega} \mid \omega \in \mathbb{R}^3 \}$ of skew-symmetric matrices is called its **Lie algebra**.

3. Consider a vector $\omega \in \mathbb{R}^3$ with $\|\omega\| = 1$ and its corresponding skew-symmetric matrix $\hat{\omega}$.

   (a) Show that $\hat{\omega}^2 = \omega \omega^\top - I$ and $\hat{\omega}^3 = -\hat{\omega}$.

   (b) Following the result of (a), find simple rules for the calculation of $\hat{\omega}^n$ and proof your result.

   Distinguish between odd and even numbers $n$.

   (c) Derive the Rodrigues’ formula for a skew-symmetric matrix $\hat{\omega}$ corresponding to an arbitrary vector $\omega \in \mathbb{R}^3$ (i.e. $\|\omega\|$ does not have to be equal to 1):

   $$ e^{\hat{\omega}} = I + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|) + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos(\|\omega\|)) $$

   *Hint:* Combine your result from (b) with

   $$ e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} \quad \text{and} \quad \sin(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \quad \text{and} \quad 1 - \cos(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^{2n}}{(2n)!} $$