Math Background

1. Show for each of the following sets (1) whether they are linearly independent, (2) whether they span \( \mathbb{R}^3 \) and (3) whether they form a basis of \( \mathbb{R}^3 \):

   (a) \( B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \end{pmatrix} \right\} \)

The set \( B_1 \) (1) is linearly independent, (2) spans \( \mathbb{R}^3 \), (3) forms a basis of \( \mathbb{R}^3 \).

This can be shown by building a matrix and calculating the determinant:

\[
\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}
\]

As the determinant is not zero, we know that the vectors are linearly independent. Three linearly independent vectors in \( \mathbb{R}^3 \) span \( \mathbb{R}^3 \). A set is a basis of \( \mathbb{R}^3 \) if it is linearly independent and spans \( \mathbb{R}^3 \), so \( B_1 \) forms a basis.

(b) \( B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ \end{pmatrix} \right\} \)

The set \( B_2 \) (1) is linearly independent, (2) does not span \( \mathbb{R}^3 \), (3) does not form a basis of \( \mathbb{R}^3 \).

Since the two vectors are not parallel, linear independence is given. To span \( \mathbb{R}^3 \), there are at least three vectors needed. Hence, the set cannot be a basis either.

(c) \( B_3 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \)

The set \( B_3 \) (1) is not linearly independent, (2) spans \( \mathbb{R}^3 \), (3) does not form a basis of \( \mathbb{R}^3 \).

In \( \mathbb{R}^3 \), there cannot be more than three independent vectors. Using e.g. the determinant, one finds that any three of the four vectors form a basis of \( \mathbb{R}^3 \) and thus the four together span \( \mathbb{R}^3 \). Since they are not linearly independent, they cannot form a basis.

2. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!

   (a) \( G_1 := \left\{ A \in \mathbb{R}^{n \times n} \mid \det(A) \neq 0 \land A^\top = A \right\} \)

The set is not closed under multiplication, thus no group. To show this, one counter-example is enough: choose \( n = 3 \) and

\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{pmatrix} \in G_1, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in G_1 : \quad AB = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 0 & 12 \\ 3 & 8 & 15 \end{pmatrix} \notin G_1
\]
Note:
You can also show that if $G_1$ was a group, for any $A, B \in G_1$, $(AB)^\top = AB$ would have to be true, but is not. This is equivalent to saying $BA = AB$ would have to be true:

$$(AB)^\top = B^\top A^\top = BA$$

However, to show that there exist $A$ and $B$ in $G_1$ for which $AB \neq BA$ (which is an important step in the proof!), the easiest way again is to choose a concrete counter-example.

(b) $G_2 := \{A \in \mathbb{R}^{n \times n} | \det(A) = -1\}$

The set contains no neutral element, thus no group:

$$\det(\text{Id}_n) = 1 \neq -1 \implies \text{Id}_n \notin G_2$$

(c) $G_3 := \{A \in \mathbb{R}^{n \times n} | \det(A) > 0\}$

The set forms a group. The easiest way to show this is to show that $G_3$ is a subgroup of the general linear group $GL(n)$. We simply need to show that for any two elements $A, B$ of $G_3$, $AB^{-1}$ is also in $G_3$:

$$\det(AB^{-1}) = \det(A)\det(B)^{-1} > 0 \iff AB^{-1} \in G_3$$

Thus, $G_3$ is a subgroup of $GL(n)$ and hence a group.

3. Prove or disprove: There exist vectors $v_1, \ldots, v_5 \in \mathbb{R}^3 \setminus \{0\}$, which are pairwise orthogonal, i.e.

$$\forall i, j = 1, \ldots, 5 : \ i \neq j \implies \langle v_i, v_j \rangle = 0$$

Assume there exist five pairwise orthogonal, non-zero vectors $v_1, \ldots, v_5 \in \mathbb{R}^3$. In $\mathbb{R}^3$, there are at most three linearly independent vectors. Thus, the vectors are linearly dependent, which means

$$\exists a_i : \sum_{i=1}^{5} a_i v_i = 0 ,$$

with at least one $a_i \neq 0$. Without loss of generality, assume that $a_1 = -1$, resulting in

$$v_1 = a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5$$

As the vectors are assumed to be pairwise orthogonal, we can derive

$$||v_1||^2 = \langle v_1, v_1 \rangle = \langle v_1, a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 \rangle = a_2 \langle v_1, v_2 \rangle + a_3 \langle v_1, v_3 \rangle + a_4 \langle v_1, v_4 \rangle + a_5 \langle v_1, v_5 \rangle = 0 + 0 + 0 + 0 = 0 \implies v_1 = 0 ,$$

which contradicts the assumption of pairwise orthogonal, non-zero vectors.

4. Groups and inclusions:

Groups

\footnote{See e.g. \url{https://en.wikipedia.org/wiki/Subgroup_test} for a proof if this is not clear to you.}
(a) $SO(n)$: special orthogonal group
(b) $O(n)$: orthogonal group
(c) $GL(n)$: general linear group
(d) $SL(n)$: special linear group
(e) $SE(n)$: special euclidean group (In particular, $SE(3)$ represents the rigid-body motions in $\mathbb{R}^3$)
(f) $E(n)$: euclidean group
(g) $A(n)$: affine group

Inclusions
(a) $SO(n) \subset O(n) \subset GL(n)$
(b) $SE(n) \subset E(n) \subset A(n) \subset GL(n+1)$

5. $\lambda_a = \frac{\langle v_a, v_b \rangle}{\langle v_a, v_a \rangle} = \frac{v_a^T A v_b}{v_a^T v_a} = \frac{v_a^T (\lambda_b v_b)}{v_a^T v_a} = \lambda_b$

6. Let $V$ be the orthonormal matrix (i.e. $V^T = V^{-1}$) given by the eigenvectors, and $\Sigma$ the diagonal matrix containing the eigenvalues:

$$V = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \ddots & 0 \\ \vdots & \ddots & \lambda_n \end{pmatrix}.$$ 

As $V$ is a basis, we can express $x$ as a linear combination of the eigenvectors $x = V \alpha$, for some $\alpha \in \mathbb{R}^n$. For $||x|| = 1$ we have $\sum \alpha_i^2 = \alpha^T \alpha = x^T V V^T x = x^T x = 1$. This gives

$$x^T Ax = x^T V \Sigma V^{-1} x = \alpha^T V V^T V \alpha = \alpha^T \Sigma \alpha = \sum \alpha_i^2 \lambda_i$$

Considering $\sum \alpha_i^2 = 1$, we can conclude that this expression is minimized iff only the $\alpha_i$ corresponding to the smallest eigenvalue(s) are non-zero. If $\lambda_{n-1} \geq \lambda_n$, there exist only two solutions ($\alpha_n = \pm 1$), otherwise infinitely many.

For maximisation, only the the $\alpha_i$ corresponding to the largest eigenvalue(s) can be non-zero.

7. We show that: $x \in \ker(A) \iff x \in \ker(A^T A)$.

"$\Rightarrow$": Let $x \in \ker(A)$

$$A^T A x \underbrace{=}_{=0} 0 \quad \Rightarrow x \in \ker(A^T A)$$

"$\Leftarrow$": Let $x \in \ker(A^T A)$

$$0 = x^T A^T A x = \langle Ax, Ax \rangle = ||Ax||^2 \quad \Rightarrow Ax = 0 \quad \Rightarrow x \in \ker(A)$$
8. Singular Value Decomposition (SVD)

Note: There exist multiple slightly different definitions of the SVD. Depending on the convention used, we might have \( S \in \mathbb{R}^{m \times n}, S \in \mathbb{R}^{n \times n}, \) or \( S \in \mathbb{R}^{p \times p} \) where \( p = \text{rank}(A) \). In the lecture the third option was presented, for which \( S \) is invertible (no zeros on the diagonal). In the following, we present the results for the first option, since that is the one that Matlab’s \texttt{svd} function returns by default.

(a) \( A \in \mathbb{R}^{m \times n}, U \in \mathbb{R}^{m \times m}, S \in \mathbb{R}^{m \times n}, V \in \mathbb{R}^{n \times n} \)

(b) Similarities and differences between SVD and EVD:
   i. Both are matrix diagonalization techniques.
   ii. The SVD can be applied to matrices \( A \in \mathbb{R}^{m \times n} \) with \( m \neq n \), whereas the EVD is only applicable to quadratic matrices \( (A \in \mathbb{R}^{m \times n} \) with \( m = n \)).

(c) Relationship between \( U, S, V \) and the eigenvalues and eigenvectors of \( A^\top A \) and \( AA^\top \):
   i. \( A^\top A \): The columns of \( V \) are eigenvectors; the squares of the diagonal elements of \( S \) are eigenvalues.
   ii. \( AA^\top \): The columns of \( U \) are eigenvectors; the squares of the diagonal elements of \( S \) are eigenvalues (possibly filled up with zeros).

(d) Entries in \( S \):
   i. \( S \) is a diagonal matrix. The elements along the diagonal are the singular values of \( A \).
   ii. The number of non-zero singular values gives us the rank of the matrix \( A \).