Multiple View Geometry: Exercise 1

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Math Background

1. Show for each of the following sets (1) whether they are linearly independent, (2) whether they span $\mathbb{R}^3$ and (3) whether they form a basis of $\mathbb{R}^3$:

   (a) $B_1 = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

   The set B_1 (1) is linearly independent, (2) spans $\mathbb{R}^3$, (3) forms a basis of $\mathbb{R}^3$.
   This can be shown by building a matrix and calculating the determinant:

   $$\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = 1 \neq 0\,.$$

   As the determinant is not zero, we know that the vectors are linearly independent. Three linearly independent vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$. A set is a basis of $\mathbb{R}^3$ if it is linearly independent and spans $\mathbb{R}^3$, so B_1 forms a basis.

   (b) $B_2 = \left\{ \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$

   The set B_2 (1) is linearly independent, (2) does not span $\mathbb{R}^3$, (3) does not form a basis of $\mathbb{R}^3$.
   Since the two vectors are not parallel, linear independence is given. To span $\mathbb{R}^3$, there are at least three vectors needed. Hence, the set cannot be a basis either.

   (c) $B_3 = \left\{ \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$

   The set B_3 (1) is not linearly independent, (2) spans $\mathbb{R}^3$, (3) does not form a basis of $\mathbb{R}^3$.
   In $\mathbb{R}^3$, there cannot be more than three independent vectors. Using e.g. the determinant, one finds that any three of the four vectors form a basis of $\mathbb{R}^3$ and thus the four together span $\mathbb{R}^3$. Since they are not linearly independent, they cannot form a basis.

2. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!

   (a) $G_1 := \left\{ A \in \mathbb{R}^{n\times n} \,|\, \det(A) \neq 0 \wedge A^\top = A \right\}$

   The set is not closed under multiplication, thus no group. To show this, one counter-example is enough: choose $n = 3$ and

   $$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{pmatrix} \in G_1\,, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in G_1\;: \quad AB = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 0 & 12 \\ 3 & 8 & 15 \end{pmatrix} \notin G_1$$

Note:
You can also show that if G_1 was a group, for any $A, B \in G_1$, $(AB)^\top = AB$ would have to be true, but is not. This is equivalent to saying $BA = AB$ would have to be true:

$$(AB)^\top = B^\top A^\top = BA$$

However, to show that there exist A and B in G_1 for which $AB \neq BA$ (which is an important step in the proof!), the easiest way again is to choose a concrete counter-example.

(b) $G_2 := \{A \in \mathbb{R}^{n \times n} | \det(A) = -1\}$
The set contains no neutral element, thus no group:

$$\det(\mathrm{Id}_n) = 1 \neq -1 \quad \Rightarrow \quad \mathrm{Id}_n \notin G_2$$

(c) $G_3 := \{A \in \mathbb{R}^{n \times n} | \det(A) > 0\}$
The set forms a group. The easiest way to show this is to show that G_3 is a subgroup of the general linear group $GL(n)$. We simply need to show that for any two elements A, B of G_3, AB^{-1} is also in G_3: [1] for $A, B \in G_3$,

$$\det(AB^{-1}) = \underbrace{\det(A)}_{>0} \underbrace{[\det(B)]^{-1}}_{>0} > 0 \quad \Rightarrow \quad AB^{-1} \in G_3$$

Thus, G_3 is a subgroup of $GL(n)$ and hence a group.

3. Prove or disprove: There exist vectors $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, which are pairwise orthogonal, i.e.

$$\forall i, j = 1, ..., 5 : \quad i \neq j \implies \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$$

Assume there exist five pairwise orthogonal, non-zero vectors $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^3$. In $\mathbb{R}^3$, there are at most three linearly independent vectors. Thus, the vectors are linearly dependent, which means

$$\exists\, a_i : \quad \sum_{i=1}^{5} a_i \mathbf{v}_i = 0\,,$$

with at least one $a_i \neq 0$. Without loss of generality, assume that $a_1 = -1$, resulting in

$$\mathbf{v}_1 = a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5$$

As the vectors are assumed to be pairwise orthogonal, we can derive

$$\begin{aligned} ||\mathbf{v}_1||^2 &= \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \\ &= \langle \mathbf{v}_1, a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 \rangle = \\ &= a_2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle + a_3\langle \mathbf{v}_1, \mathbf{v}_3 \rangle + a_4\langle \mathbf{v}_1, \mathbf{v}_4 \rangle + a_5\langle \mathbf{v}_1, \mathbf{v}_5 \rangle = \\ &= 0 + 0 + 0 + 0 = 0 \\ \Rightarrow \quad \mathbf{v}_1 &= \mathbf{0}\,, \end{aligned}$$

which contradicts the assumption of pairwise orthogonal, non-zero vectors.

4. Groups and inclusions:
Groups

[1] See e.g. `https://en.wikipedia.org/wiki/Subgroup_test` for a proof if this is not clear to you.

(a) $SO(n)$: special orthogonal group

(b) $O(n)$: orthogonal group

(c) $GL(n)$: general linear group

(d) $SL(n)$: special linear group

(e) $SE(n)$: special euclidean group (In particular, $SE(3)$ represents the rigid-body motions in $\mathbb{R}^3$)

(f) $E(n)$: euclidean group

(g) $A(n)$: affine group

Inclusions

(a) $SO(n) \subset O(n) \subset GL(n)$

(b) $SE(n) \subset E(n) \subset A(n) \subset GL(n+1)$

5. $\lambda_a = \frac{(\lambda_a v_a)^\top v_b}{\langle v_a, v_b \rangle} = \frac{v_a^\top A^\top v_b}{\langle v_a, v_b \rangle} = \frac{v_a^\top A v_b}{\langle v_a, v_b \rangle} = \frac{v_a^\top (\lambda_b v_b)}{\langle v_a, v_b \rangle} = \lambda_b$

6. Let V be the orthonormal matrix (i.e. $V^\top = V^{-1}$) given by the eigenvectors, and Σ the diagonal matrix containing the eigenvalues:

$$V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \lambda_1 & 0 & \ddots \\ 0 & \ddots & 0 \\ \ddots & 0 & \lambda_n \end{pmatrix}.$$

As V is a basis, we can express x as a linear combination of the eigenvectors $x = V\alpha$, for some $\alpha \in \mathbb{R}^n$. For $||x|| = 1$ we have $\sum_i \alpha_i^2 = \alpha^\top \alpha = x^\top V V^\top x = x^\top x = 1$. This gives

$$\begin{aligned} x^\top A x &= x^\top V \Sigma V^{-1} x \\ &= \alpha^\top V^\top V \Sigma V^\top V \alpha \\ &= \alpha^\top \Sigma \alpha = \sum_i \alpha_i^2 \lambda_i \end{aligned}$$

Considering $\sum_i \alpha_i^2 = 1$, we can conclude that this expression is minimized iff only the α_i corresponding to the smallest eigenvalue(s) are non-zero. If $\lambda_{n-1} \gneq \lambda_n$, there exist only two solutions ($\alpha_n = \pm 1$), otherwise infinitely many.

For maximisation, only the the α_i corresponding to the largest eigenvalue(s) can be non-zero.

7. We show that: $x \in \text{kernel}(A) \Leftrightarrow x \in \text{kernel}(A^\top A)$.

"$\Rightarrow$": Let $x \in \text{kernel}(A)$

$$A^\top \underbrace{Ax}_{=0} = A^\top 0 = 0 \quad \Rightarrow x \in \text{kernel}(A^\top A)$$

"$\Leftarrow$": Let $x \in \text{kernel}(A^\top A)$

$$0 = x^\top \underbrace{A^\top A x}_{=0} = \langle Ax, Ax \rangle = ||Ax||^2 \quad \Rightarrow Ax = 0 \quad \Rightarrow x \in \text{kernel}(A)$$

8. Singular Value Decomposition (SVD)
   Note: There exist multiple slightly different definitions of the SVD. Depending on the convention used, we might have $S \in \mathbb{R}^{m \times n}$, $S \in \mathbb{R}^{n \times n}$, or $S \in \mathbb{R}^{p \times p}$ where $p = \mathrm{rank}(A)$. In the lecture the third option was presented, for which S is invertible (no zeros on the diagonal). In the following, we present the results for the first option, since that is the one that Matlab's `svd` function returns by default.

   (a) $A \in \mathbb{R}^{m \times n}, U \in \mathbb{R}^{m \times m}, S \in \mathbb{R}^{m \times n}, V \in \mathbb{R}^{n \times n}$

   (b) Similarities and differences between SVD and EVD:

      i. Both are matrix diagonalization techniques.

      ii. The SVD can be applied to matrices $A \in \mathbb{R}^{m \times n}$ with $m \neq n$, whereas the EVD is only applicable to quadratic matrices ($A \in \mathbb{R}^{m \times n}$ with $m = n$).

   (c) Relationship between U, S, V and the eigenvalues and eigenvectors of $A^\top A$ and $AA^\top$:

      i. $A^\top A$: The columns of V are eigenvectors; the squares of the diagonal elements of S are eigenvalues.

      ii. $AA^\top$: The columns of U are eigenvectors; the squares of the diagonal elements of S are eigenvalues (possibly filled up with zeros).

   (d) Entries in S:

      i. S is a diagonal matrix. The elements along the diagonal are the *singular values* of A.

      ii. The number of non-zero singular values gives us the *rank* of the matrix A.