Multiple View Geometry: Solution 4

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`Wednesdays 16:00-18:15 at Hörsaal 2, "Interims I" (5620.01.102), and on RBG Live`

Exercise: May 31, 2023

The Lucas-Kanade method

1. Prove that the minimizer $\mathbf{b}$ of $E(\mathbf{v})$ can be written as

$$\mathbf{b} = -M^{-1}\mathbf{q}$$

where the entries of M and $\mathbf{q}$ are given by

$$m_{ij} = G * \left(I_{x_i} \cdot I_{x_j}\right) \quad \text{and} \quad q_i = G * (I_{x_i} \cdot I_t)$$

We start by expanding the squared term in the energy $E(\mathbf{v})$:

$$E(\mathbf{v}) = \int_{W(\mathbf{x})} G(\mathbf{x}-\mathbf{x}')\left(\nabla I(\mathbf{x}',t)^\top \mathbf{v}\right)^2 d\mathbf{x}' + \int_{W(\mathbf{x})} G(\mathbf{x}-\mathbf{x}') 2\nabla I(\mathbf{x}',t)^\top \mathbf{v}\partial_t I(\mathbf{x}',t) d\mathbf{x}' + \\ + \int_{W(\mathbf{x})} G(\mathbf{x}-\mathbf{x}')\left(\partial_t I(\mathbf{x}',t)\right)^2 d\mathbf{x}'$$

Now, for each term in the sum we take the derivative (gradient) with respect to $\mathbf{v}$:

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}\mathbf{v}} &= \int_{W(\mathbf{x})} G(\mathbf{x}-\mathbf{x}') 2\nabla I(\mathbf{x}',t)\left(\nabla I(\mathbf{x}',t)^\top \mathbf{v}\right) d\mathbf{x}' + \\ &\quad + \int_{W(\mathbf{x})} G(\mathbf{x}-\mathbf{x}') 2\nabla I(\mathbf{x}',t)\partial_t I(\mathbf{x}',t) d\mathbf{x}' + 0 = \\ &= 2\left(G * \left(\nabla I \nabla I^\top\right)\right)\mathbf{v} + 2\left(G * (\nabla I \partial_t I)\right) =: 2M\mathbf{v} + 2\mathbf{q}\end{aligned}$$

where M is defined as $G * \left(\nabla I \nabla I^\top\right)$ and $\mathbf{q}$ as $G * (\nabla I \partial_t I)$. We further know $\nabla I \nabla I^\top$ and $\nabla I \partial_t I$:

$$\nabla I \nabla I^\top = \begin{pmatrix} I_x \\ I_y \end{pmatrix}\begin{pmatrix} I_x & I_y \end{pmatrix} = \begin{pmatrix} (I_x)^2 & I_x I_y \\ I_x I_y & (I_y)^2 \end{pmatrix} \quad \text{and} \quad \nabla I \partial_t I = \begin{pmatrix} I_x I_t \\ I_y I_t \end{pmatrix}$$

which proves that the entries of M and $\mathbf{q}$ are as stated. Since we want to find a minimizer $\mathbf{b}$ of $E(\mathbf{v})$, we require

$$\left.\frac{\mathrm{d}E(\mathbf{v})}{\mathrm{d}\mathbf{v}}\right|_{\mathbf{v}=\mathbf{b}} = 0 \quad \Rightarrow \quad 2M\mathbf{b} + 2\mathbf{q} = 0 \quad \Rightarrow \quad \mathbf{b} = -M^{-1}\mathbf{q}$$

2. Show that if the gradient direction is constant in $W(\mathbf{x})$, i.e. $\nabla I(\mathbf{x}',t) = \alpha(\mathbf{x}',t)\mathbf{u}$ for a scalar function α and a 2D vector $\mathbf{u}$, M is not invertible.
$\mathbf{u}$ does not depend on $\mathbf{x}'$, so it can be pulled out of the convolution integral. Thus,

$$M = G * \left(\nabla I \nabla I^\top\right) = \left(G * \alpha^2\right)\mathbf{u}\mathbf{u}^\top \quad \Rightarrow \quad \det M = \left(G * \alpha^2\right)^2 \left(u_1^2 u_2^2 - (u_1 u_2)^2\right) = 0\,.$$

Explain how this observation is related to the aperture problem.

The aperture problem states that it is impossible to determine the motion orthogonal to the gradient direction in regions with constant gradient direction,. M not being invertible means that there is no unique solution $\mathbf{b}$, which is the mathematical formulation of "the motion cannot be determined".

3. Write down explicit expressions for the two components b_1 and b_2 of the minimizer in terms of m_{ij} and q_i.

$$\mathbf{b} = -M^{-1}\mathbf{q} \quad \text{where} \quad M^{-1} = \frac{1}{\text{det}M}\begin{pmatrix} m_{22} & -m_{12} \\ -m_{12} & m_{11} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{-1}{m_{11}m_{22} - m_{12}^2}\begin{pmatrix} m_{22} & -m_{12} \\ -m_{12} & m_{11} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{m_{22}q_1 - m_{12}q_2}{m_{12}^2 - m_{11}m_{22}} \\ \\ \frac{m_{11}q_2 - m_{12}q_1}{m_{12}^2 - m_{11}m_{22}} \end{pmatrix}$$