Multiple View Geometry: Solution 5

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1. (a) E is essential matrix $\quad\Rightarrow\quad \Sigma = \text{diag}\{\sigma, \sigma, 0\}$:

$$R_z(\pm\frac{\pi}{2})\Sigma = \begin{pmatrix} 0 & \mp 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mp\sigma & 0 \\ \pm\sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -(R_z(\pm\frac{\pi}{2})\Sigma)^\top$$

$$\begin{aligned} -\hat{T}^\top &= -(UR_z\Sigma U^\top)^\top \\ &= U(-R_z\Sigma)^\top U^\top \\ &= UR_z\Sigma U^\top \\ &= \hat{T} \end{aligned}$$

(b) Since U, V are orthogonal with determinant 1, they are rotation matrices. Since $\text{SO}(3)$ is a group and thus closed under multiplication, $R \in \text{SO}(3)$.

Alternative longer proof:

i. U, V are orthogonal matrices $\Rightarrow U^\top U = \mathbb{1}$ and $VV^\top = \mathbb{1}$
R_z is a rotation matrix $\Rightarrow R_z R_z^\top = \mathbb{1}$

$$\begin{aligned} R^\top R &= (UR_z^\top V^\top)^\top (UR_z^\top V^\top) \\ &= VR_z U^\top U R_z^\top V^\top \\ &= VR_z R_z^\top V^\top \\ &= VV^\top \\ &= \mathbb{1} \end{aligned}$$

ii. U and V are special orthogonal matrices with $\det(U) = \det(V^\top) = 1$.

$$\det(R) = \det(UR_z^\top V^\top) = \underbrace{\det(U)}_{1} \cdot \underbrace{\det(R_z^\top)}_{1} \cdot \underbrace{\det(V^\top)}_{1} = 1$$

2. (a) $H = R + Tu^\top \Leftrightarrow R = H - Tu^\top$.

$$\begin{aligned} E &= \hat{T}R \\ &= \hat{T}(H - Tu^\top) \\ &= \hat{T}H - \underbrace{\hat{T}T}_{=T\times T=0} u^\top \\ &= \hat{T}H \end{aligned}$$

(b)

$$\begin{aligned}
H^\top E + E^\top H &= H^\top(\hat{T}H) + (\hat{T}H)^\top H \\
&= H^\top(\hat{T}H) + H^\top \hat{T}^\top H \\
&= H^\top \hat{T} H - H^\top \hat{T} H \quad \text{(because } \hat{T} \text{ is skew-symmetric, i.e. } \hat{T}^\top = -\hat{T}\text{)} \\
&= 0
\end{aligned}$$

3. In this exercise we assume pixel coordinates.

Rotation R and translation T are defined such that

$$g_{21} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$$

transforms a point from coordinate system 1 (CS1) to coordinate system 2 (CS2). This means that the inverse transformation (converting points from CS2 to CS1) is given by

$$g_{12} = g_{21}^{-1} = \begin{bmatrix} R^\top & -R^\top T \\ 0 & 1 \end{bmatrix}.$$

o_1 seen in CS1: $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top$ (homogeneous coordinates)

o_1 seen in CS2: $g_{21} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top = \begin{bmatrix} T \\ 1 \end{bmatrix}$

e_2 are the pixel coordinates of o_1 projected into image 2:

$$\lambda_2 e_2 = K_2 \Pi_0 \begin{bmatrix} T \\ 1 \end{bmatrix} = K_2 T$$

o_2 seen in CS2: $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top$

o_2 seen in CS1: $g_{12} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top = \begin{bmatrix} -R^\top T \\ 1 \end{bmatrix}$

e_1 are the pixel coordinates of o_2 projected into image 1:

$$\lambda_1 e_1 = K_1 \Pi_0 \begin{bmatrix} -R^\top T \\ 1 \end{bmatrix} = -K_1 R^\top T$$

$$\begin{aligned}
F e_1 &= \underbrace{(K_2^{-\top} \hat{T} R K_1^{-1})}_{F} \underbrace{(-\frac{1}{\lambda_1} K_1 R^\top T)}_{e_1} \\
&= -\frac{1}{\lambda_1} K_2^{-\top} \hat{T} R \underbrace{K_1^{-1} K_1}_{\mathbb{1}} R^\top T \\
&= -\frac{1}{\lambda_1} K_2^{-\top} \hat{T} \underbrace{R R^\top}_{\mathbb{1}} T \\
&= -\frac{1}{\lambda_1} K_2^{-\top} \underbrace{\hat{T} T}_{= T \times T = 0} \\
&= 0
\end{aligned}$$

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\begin{aligned}
e_2^\top F &= (\underbrace{\frac{1}{\lambda_2} K_2 T}_{e_2})^\top (\underbrace{K_2^{-\top} \hat{T} R K_1^{-1}}_{F}) \\
&= \frac{1}{\lambda_2} T^\top \underbrace{K_2^\top K_2^{-\top}}_{\mathbb{1}} \hat{T} R K_1^{-1} \\
&= \frac{1}{\lambda_2} T^\top \hat{T} R K_1^{-1} \\
&= \frac{1}{\lambda_2} (\hat{T}^\top T)^\top R K_1^{-1} \\
&= \frac{1}{\lambda_2} (-\hat{T} T)^\top R K_1^{-1} \\
&= -\frac{1}{\lambda_2} (T \times T)^\top R K_1^{-1} \\
&= -\frac{1}{\lambda_2} 0 R K_1^{-1} \\
&= 0
\end{aligned}
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