

Practical Course: Vision Based Navigation

Lecture 2: Camera Models and Optimization

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Camera Models

Image Formation





Image Formation





Image Formation





Camera Obscura





First published picture of camera obscura in Gemma Frisius' 1545 book De Radio Astronomica et Geometrica

Pinhole Camera Model



- Camera coordinate frame attached to the center of (0,0) pixel.
 - X horizontal axis
 - Y vertical axis downwards
 - Z forward

Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y \right]^T$$

Projection:

$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \frac{x}{z} \\ f_y \frac{y}{z} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix}$$

$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \frac{1}{\sqrt{m_x^2 + m_y^2 + 1}} \begin{bmatrix} m_x \\ m_y \\ m_y \end{bmatrix}$$
$$m_x = \frac{u - c_x}{f_x},$$
$$m_y = \frac{v - c_y}{f_y}.$$



Top View



Optical Lens: Distortion and Aperture



Blue: perfect focus, Green: less for Red: out of focus [source: unrealengine 5.0 docs]

Large FOV and Navigation





Z. Zhang, H. Rebecq, C. Forster, D. Scaramuzza "**Benefit of Large Field-of-View Cameras for Visual Odometry**" IEEE International Conference on Robotics and Automation (ICRA), Stockholm, 2016.

Distortion

Pinhole-Undistorted



- Pinhole
 - Fast projection and unprojection
 - Not suitable for > 180°
 - Bad numeric properties > 120°

Original Image



- More complex model
 - Working with "raw" image
 - No issues with large FOV
 - Possible to optimize intrinsics
 online

(Extended) Unified Camera Model

Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y, \alpha, \beta \right]^T$$

Projection:

$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \frac{x}{\alpha d + (1 - \alpha)z} \\ f_y \frac{y}{\alpha d + (1 - \alpha)z} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix},$$
$$d = \sqrt{\beta(x^2 + y^2) + z^2}.$$

$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \frac{1}{\sqrt{m_x^2 + m_y^2 + m_z^2}} \begin{bmatrix} m_x \\ m_y \\ m_y \\ m_z \end{bmatrix},$$

$$m_x = \frac{u - c_x}{f_x},$$

$$m_y = \frac{v - c_y}{f_y},$$

$$r^2 = m_x^2 + m_y^2,$$

$$m_z = \frac{1 - \beta \alpha^2 r^2}{\alpha \sqrt{1 - (2\alpha - 1)\beta r^2} + (1 - \alpha)},$$

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Kannala-Brandt Camera Model

Intrinsic parameters:

$$\mathbf{i} = [f_x, f_y, c_x, c_y, k_1, k_2, k_3, k_4]^T$$

Projection:

$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \ d(\theta) \ \frac{x}{r} \\ f_y \ d(\theta) \ \frac{y}{r} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix},$$
$$r = \sqrt{x^2 + y^2},$$
$$\theta = \operatorname{atan2}(r, z),$$
$$d(\theta) = \theta + k_1 \theta^3 + k_2 \theta^5 + k_3 \theta^7 + k_4 \theta^9.$$

$$\pi^{-1}(\mathbf{u}, \mathbf{i}) = \begin{bmatrix} \sin(\theta^*) \frac{m_x}{r_u} \\ \sin(\theta^*) \frac{m_y}{r_u} \\ \cos(\theta^*) \end{bmatrix},$$
$$m_x = \frac{u - c_x}{f_x},$$
$$m_y = \frac{v - c_y}{f_y},$$
$$r_u = \sqrt{m_x^2 + m_y^2},$$
$$\theta^* = d^{-1}(r_u),$$

Double Sphere Camera Model

Intrinsic parameters:

$$\mathbf{i} = \left[f_x, f_y, c_x, c_y, \xi, \alpha\right]^T$$

Projection:

$$\pi(\mathbf{x}, \mathbf{i}) = \begin{bmatrix} f_x \frac{x}{\alpha d_2 + (1 - \alpha)(\xi d_1 + z)} \\ f_y \frac{y}{\alpha d_2 + (1 - \alpha)(\xi d_1 + z)} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix},$$
$$d_1 = \sqrt{x^2 + y^2 + z^2},$$
$$d_2 = \sqrt{x^2 + y^2 + (\xi d_1 + z)^2}.$$

$$\begin{aligned} \pi^{-1}(\mathbf{u},\mathbf{i}) &= \frac{m_z \xi + \sqrt{m_z^2 + (1 - \xi^2) r^2}}{m_z^2 + r^2} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \xi \end{bmatrix}, \\ m_x &= \frac{m_z c_x}{f_x}, \\ m_y &= \frac{v - c_y}{f_y}, \\ r^2 &= m_x^2 + m_y^2, \\ m_z &= \frac{1 - \alpha^2 r^2}{\alpha \sqrt{1 - (2\alpha - 1)r^2} + 1 - \alpha}. \end{aligned}$$

The Double Sphere Camera Model (V. Usenko, N. Demmel and D. Cremers), *In Proc. of the Int. Conference on 3D Vision (3DV)*, 2018. [arXiv:1807.08957]

Camera Models Code

```
template <typename Scalar>
class PinholeCamera : public AbstractCamera<Scalar> {
public:
 •••
 typedef Eigen::Matrix<Scalar, 2, 1> Vec2;
 typedef Eigen::Matrix<Scalar, 3, 1> Vec3;
 typedef Eigen::Matrix<Scalar, N, 1> VecN;
 PinholeCamera() { param.setZero(); }
 PinholeCamera(const VecN& p) { param = p; }
 virtual Vec2 project(const Vec3& p) const {
    const Scalar& fx = param[0];
    const Scalar& fy = param[1];
    const Scalar& cx = param[2];
   const Scalar& cy = param[3];
    const Scalar& x = p[0];
    const Scalar& y = p[1];
    const Scalar& z = p[2];
   Vec2 res;
    // TODO SHEET 2: implement camera model
   return res;
 }
 virtual Vec3 unproject(const Vec2& p) const {
    const Scalar& fx = param[0];
    const Scalar& fy = param[1];
    const Scalar& cx = param[2];
    const Scalar& cy = param[3];
   Vec3 res;
   // TODO SHEET 2: implement camera model
    return res;
 }
 EIGEN MAKE ALIGNED OPERATOR NEW
private:
 VecN param;
```

- Avoid using std::pow() function to maintain the precision. For example, if you need to compute x² use multiplication: x * x.
- If your compiler complains about Jet types try changing the constants in projection and unprojection functions to Scalar(<constant>). For example, Scalar(1) instead of 1.
- You can use <u>Newton's method for finding roots</u> to compute a root of the polynomial given a good initialization. Usually 3-5 iterations should be enough for the optimization to converge.
- You can use <u>Horner's method</u> to efficiently compute polynomials.

Optimization

Given a set of parameters $x = \{x_1, \dots, x_n\}$ and a set of observations that depend on the parameters $z = \{z_1, \ldots, z_m\}$ we want estimate the value of x that is most likely to result in these observations:

$$x^* = \operatorname{argmax}_{r} P(x \,|\, z),$$

This estimate of the parameters x^* is called the Maximum a posteriori (MAP) estimation.

We can rewrite the probability using the Bayes' Rule:

Posteriori Likelihood Prior
$$P(x \mid z) = \frac{P(z \mid x)P(x)}{P(z)}.$$

We can drop the denominator, because it does not depend on x.

$$x^* = \underset{x}{\operatorname{argmax}} P(z \mid x)P(x).$$

"Which state it is most likely to produce such measurements?"

- From MAP to least squares problem
- If we assume that the measurements are independent the joint PDF can be factorized:

$$P(z \mid x) = \prod_{k=0}^{K} P(z_k \mid x)$$

- Let's consider a single observation: $z_k = h(x) + v_k$,
 - Affected by Gaussian noise: $v_k \sim N(0, Q_k)$
- The observation model gives us a conditional PDF:

$$P(z_k \mid x) = N(h(x), Q_k)$$

• How do we estimate *x* ?

• Gaussian Distribution (matrix form)

$$P(x) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right).$$

• Take negative logarithm from both sides:

$$-\ln P(x) = \frac{1}{2} \ln((2\pi)^p |\Sigma|) + \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu).$$

• Maximum of P(x) is equivalent to the minimum of $-\ln P(x)$.

- Batch least squares
- Formulate residual function:

$$r_k = z_k - h(x).$$

• Maximizing of P(x) is equivalent to the minimizing the sum of squared residuals:

$$E(x) = \frac{1}{2} \sum_{k} r_k^T Q r_k.$$

• Some notes:

- Because of noise, when we take the estimated trajectory and map into the models, they won't fit perfectly
- Then we adjust our estimation to get a better estimation (minimize the error)
- The error distribution is affected by noise distribution (information matrix)
- Structure of the least square problem
 - Sum of many squared errors
 - The dimension of total state variable may be high
 - But single error item is easy (only related to two states in our case)
 - If we use Lie group and Lie algebra, then it's a non-constrained least square

$$E(x) = \frac{1}{2} \sum_{k} r_k^T Q r_k$$

Least Squares

 $\frac{\partial E(x)}{\partial x} = 0$

 ∂x

- How to solve a least square problem?
 - Non-linear, discrete time, non-constrained
- Let's start from a simple example
 - Consider minimizing a squared error:
 - When E(x) is simple, just solve:

$$E(x) = \frac{1}{2} \sum_{k} r_k(x)^T r_k(x) = \frac{1}{2} r(x)^T r(x)$$

Least Squares

- When E(x) is a complicated function: $\partial E(x)$
 - $\frac{\partial E(x)}{\partial x} = 0$ is hard to solve
 - We use iterative methods
- Iterative methods
 - 1. Start from an initial estimate x_0
 - 2. At iteration *n*, we find an increment Δx_n that minimizes $E(x_n + \Delta x_n)$.
 - 3. If the change in error function is small enough, stop (converged).
 - 4. If not, set $x_{n+1} = x_n + \Delta x_n$ and iterate to step 2.

Gradient Descent

• How to find the increment?

- First order methods Gradient Descent
 - Taylor expansion of the objective function
 - $E(x + \Delta x) = E(x) + G(x)\Delta x$

The update step:

 $\Delta x = -\alpha G(x)$

Gradient Descent Performance

Zig-zag in steepest descent:

- Other shortcomings:
 - Slow convergence speed
 - Even slower when close to minimum

Second Order Methods

- Second order methods
 - Taylor expansion of the objective function
 - $E(x + \Delta x) = E(x) + G(x)\Delta x + \Delta x^T H(x)\Delta x$

. Setting
$$\frac{\partial E(x + \Delta x)}{\partial \Delta x} = 0$$

The update step: $H(x)\Delta x = -G(x) \implies \Delta x = -H^{-1}(x) G(x)$

This is called Newton's method.

Second Order Methods for Least Squares

ТШ

- Second order method converges more quickly than first order methods
- But the Hessian matrix may be hard to compute
- Can we avoid the Hessian matrix and also keep second order's convergence speed? Yes, for least squares problems, there exists faster options:
 - Gauss-Newton
 - Levenberg-Marquardt

Gauss-Newton Method

- Gauss-Newton
 - Taylor expansion of r(x): $r(x + \Delta x) \simeq r(x) + J(x)\Delta x$
 - Then the squared error becomes:

$$E(x + \Delta x) = \frac{1}{2}r(x)^T r(x) + \Delta x^T J(x)^T r(x) + \frac{1}{2}\Delta x^T J(x)^T J(x)\Delta x$$
$$= F(x) + \Delta x^T J(x)^T r(x) + \frac{1}{2}\Delta x^T J(x)^T J(x)\Delta x$$

If we set $\frac{\partial E(x + \Delta x)}{\partial \Delta x} = 0$ we get:

 $J^{T}(x)J(x)\Delta x = -J(x)^{T}r(x) \implies \Delta x = -(J^{T}(x)J(x))^{-1} J(x)^{T}r(x)$ $\simeq H(x) \quad \text{Newton's Method} \quad \simeq G(x)$

Gauss-Newton Method

- Gauss-Newton uses $J^{T}(x)J(x)$ as an approximation of the Hessian
 - Avoids the computation of H(x) in the Newton's method
- But $J^T(x)J(x)$ is only semi-positive definite
- H(x) maybe singular when $J^{T}(x)J(x)$ has null space

Levenberg-Marquardt Method

- Trust region approach: approximation is only valid in a region
- Evaluate if the approximation is good:

$$\rho = \frac{r(x + \Delta x) - r(x)}{J(x)\Delta x}$$

Real descent/approx. descent

- If ρ is large, increase the region
- If ρ is small, decrease the region

• LM optimization:

$$E(x + \Delta x) = \frac{1}{2}r(x + \Delta x)^T r(x + \Delta x) + \lambda \|\Delta x\|^2$$

- Assume the approximation is only good within a region
- λ controls the region based on ρ

Levenberg-Marquardt Method

• Trust region problem:

$$E(x + \Delta x) = \frac{1}{2}r(x + \Delta x)^T r(x + \Delta x) + \lambda \|\Delta x\|^2$$

• Expand it just like in GN case, the incremental is:

$$\Delta x = - (J^T(x)J(x) + \lambda I)^{-1} J(x)^T r(x)$$

- The λI part makes sure that Hessian is positive definite.
- When $\lambda = 0$ LM becomes GN.
- When $\lambda \to \infty$ LM becomes gradient descent.

Other Methods

- Dog-leg method
- Conjugate gradient method
- Quasi-Newton's method
- Pseudo-Newton's method

• ...

- You can find more in optimization books if you are interested
- In SLAM/SfM/VO, Gauss-Netwton and Levenberg-Marquardt are used to solve camera motion, optical-flow, etc.

More details:

Triggs B, McLauchlan PF, Hartley RI, Fitzgibbon AW. Bundle adjustment—a modern synthesis. InInternational workshop on vision algorithms 1999 Sep 20 (pp. 298-372). Springer, Berlin, Heidelberg.

- We will use Ceres for least-squares optimization.
- Tutorial: <u>http://ceres-solver.org/tutorial.html</u>
- Curve fitting example: $y = \exp(mx + c)$
- Observations: a set of (x, y) pairs
- Parameters to estimate: *m*, *c*.

Ceres

• Define your residual class as a functor (overload the () operator)

```
struct ExponentialResidual {
    ExponentialResidual(double x, double y)
        : x_(x), y_(y) {}
    template <typename T>
    bool operator()(const T* const m, const T* const c, T* residual) const {
        residual[0] = T(y_) - exp(m[0] * T(x_) + c[0]);
        return true;
    }
    private:
    // Observations for a sample.
    const double x_;
    const double y_;
};
```

Ceres

• Build the optimization problem

```
double m = 0.0;
double c = 0.0;
Problem problem;
for (int i = 0; i < kNumObservations; ++i) {
    CostFunction* cost_function =
        new AutoDiffCostFunction<ExponentialResidual, 1, 1, 1>(
        new ExponentialResidual(data[2 * i], data[2 * i + 1]));
    problem.AddResidualBlock(cost_function, NULL, &m, &c);
}
```

• With auto-diff, Ceres will compute the Jacobians for you

Ceres

- Finally solve it by calling the Solve() function and get the result summary
- You can set some parameters like number of iterations, stop conditions or the linear solver type.

// Run the solver!

Solver::Options options; options.linear_solver_type = ceres::DENSE_QR; options.minimizer_progress_to_stdout = true; Solver::Summary summary; Solve(options, &problem, &summary);

std::cout << summary.BriefReport() << "\n"; std::cout << "m : " << m << "c : " << c << "\n";

Least-Squares Summary

- In the maximum a posteriori estimation we estimate all the state variable given using a set of noisy measurements.
- The MAP estimation problem with Gaussian noise can be reformulated into a least square problem
- It can be solved by iterative methods: Gradient Descent, Newton's method, Gauss-Newton or Levernberg-Marquardt.

Exercise 2

ТЛП

- We want to estimate
 - Poses of the camera setup with respect to pattern
 - Intrinsic parameters of both cameras
 - Extrinsic parameters (rigid body transformation from one camera to the other)
- Minimizing the projection residuals:

$$r = u_j - \pi (R_{cw} p_w^j + t_{cw}, \mathbf{i}),$$

- u_j detection of the corner j in the image.
- p_w^j 3D coordinates in the world (pattern) coordinate frame
- \boldsymbol{i} intrinsic parameter of the camera
- R_{cw} , t_{cw} rigid body transformation from the world (pattern) coordinate frame to the camera coordinate frame.
- π is the projection function
- Corner points are detected using Apriltags

E. Olson. AprilTag: A robust and flexible visual fiducial system. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA)*, pages 3400–3407. IEEE, May 2011.

Exercise 2 Residual

```
ТШП
```

```
struct ReprojectionCostFunctor {
  EIGEN MAKE ALIGNED OPERATOR NEW
  ReprojectionCostFunctor(const Eigen::Vector2d& p 2d,
                          const Eigen::Vector3d& p 3d,
                          const std::string& cam model)
      : p 2d(p 2d), p 3d(p 3d), cam model(cam model) {}
  template <class T>
  bool operator()(T const* const sT w i, T const* const sT i c,
                  T const* const sIntr, T* sResiduals) const {
   Eigen::Map<Sophus::SE3<T> const> const T w i(sT w i);
   Eigen::Map<Sophus::SE3<T> const> const T_i_c(sT_i_c);
   Eigen::Map<Eigen::Matrix<T, 2, 1>> residuals(sResiduals);
    const std::shared ptr<AbstractCamera<T>> cam =
        AbstractCamera<T>::from data(cam model, sIntr);
    // TODO SHEET 2: implement the rest of the functor
    return true;
  }
  Eigen::Vector2d p 2d;
  Eigen::Vector3d p 3d;
  std::string cam model;
};
```


Exercise 2

- Use camera models presented here to get initial projections
- Implement the projection function
- Implement the residual.
- Set up optimization problem. Use local parametrization where necessary.
- Test different models. How well do they fit the lens?