Multiple View Geometry: Exercise Sheet 1

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This exercise sheet covers the topics of groups, vector spaces and inner products. As groups are the most basic algebraic structure, we will start with them. Then we will move on to vector spaces, which fulfill the group properties and have additional properties. Finally, we will introduce inner products, which need a vector space to be defined.
Groups will be content of the lecture on Wednesday. We recommend you to also go through the first problems, as they are important for the understanding of the following exercises. But if you feel unsure, you can start with Problem 7 or Problem 8.

1. State the definition of a group
2. Let $(M, \cdot)$ be a group, with the right identity $e \in M$, $a \cdot e = a$, and the right inverse element $a^{-1} \in M$, $a \cdot a^{-1} = e$. Show that and the right identity is also the left identity and the right inverse is also the left inverse element.
3. Let $(M, \cdot)$ be a group, show that the inverse element $a^{-1} \in M$ of $a \in M$ is unique
4. Is the following statement correct? For groups, whose operation does not fulfil the commutative property (e.g. matrix multiplication) the left and a right inverse elements are distinct.
5. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!
   (a) $G_1 := \left\{A \in \mathbb{R}^{n \times n} \mid \det(A) \neq 0 \wedge A^\top = A\right\}$
   (b) $G_2 := \{A \in \mathbb{R}^{n \times n} \mid \det(A) = -1\}$
   (c) $G_3 := \{A \in \mathbb{R}^{n \times n} \mid \det(A) > 0\}$
6. Which groups have you seen in the lecture? Write down the names and the correct inclusions! (e.g.: group A $\subset$ group B)

7. State the definition of a vector space V over a field $\mathbb{K}$(which is eiher $\mathbb{C}$ or $\mathbb{K}$). Neglect the definition of a field here. Does V have to fulfil the group properties? What additional properties does a vectorspace fulfil?
8. Let V be a vector space over $\mathbb{K}$. State the definition of
   - linear independence of pairwise distinct $\mathbf{v}_1, ..., \mathbf{v}_k \in V$
   - the span of a set $M \subset V$
   - the basis of $U \subset V$.
9. Show (without using concepts like determinant) for each of the following sets (1) whether they are linearly independent, (2) whether they span $\mathbb{R}^3$ and (3) whether they form a basis of $\mathbb{R}^3$:

(a) $M_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) $M_2 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

(c) $M_3 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

10. The dimension theorem for vector spaces states: *Given a vector space V, any two bases have the same cardinality. This number defines the dimension of the vector space.*
    Show by using the the previous exercise: In $\mathbb{R}^3$, there cannot be more than three independent vectors.

11. A hilbert space H is a finite dimensional vector space over a field $\mathbb{K}$ endowed with an inner product. State the definition of an inner product.

12. State for the following, whether the following vector spaces form a Hilbert space with the provided inner product.
    - $\mathbb{R}^n$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$
    - $\mathbb{R}^{n \times m}$ with $\langle A, B \rangle = \mathrm{tr}(A^T B)$, with $\mathrm{tr}(A) = \sum_{i=1}^n a_{ii}$ for $A \in \mathbb{R}^{n \times n}$

13. Prove or disprove: There exist vectors $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, which are pairwise orthogonal, i.e.
    $$\forall i, j = 1, ..., 5 : \quad i \neq j \implies \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$$
    Hint: From the previous problem you can use: *In $\mathbb{R}^3$, there are at most three linearly independent vectors.*

14. Show that the frobenius norm $\|A\|_F = \sqrt{\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2}$ for $A \in \mathbb{R}^{n \times m}$ is an induced norm of the inner product $\langle A, B \rangle = \mathrm{tr}(A^\top B)$.