Multiple View Geometry: Exercise Sheet 3

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`Wednesdays 16:15-17:45 at Hörsaal 2, "Interims I" (5620.01.102), and on RBG Live`

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Part I: Theory

1. Write down the matrices $M \in SE(3) \subset \mathbb{R}^{4\times4}$ representing the following transformations:
   (a) Translation by the vector $T \in \mathbb{R}^3$.
   (b) Rotation by the rotation matrix $R \in \mathbb{R}^{3\times3}$.
   (c) Rotation by R followed by the translation T.
   (d) Translation by T followed by the rotation R.

2. Let $M_1, M_2 \in \mathbb{R}^{3\times3}$. Please prove the following:

$$\begin{array}{ccc} \mathbf{x}^\top M_1 \mathbf{x} = \mathbf{x}^\top M_2 \mathbf{x} & \text{iff} & M_1 - M_2 \text{ is skew-symmetric} \\ \text{for all } \mathbf{x} \in \mathbb{R}^3 & & (\text{i.e. } M_1 - M_2 \in so(3)) \end{array}$$

   Info: The group $SO(3)$ is called a **Lie group**.
   The space $so(3) = \{\hat{\omega} \mid \omega \in \mathbb{R}^3\}$ of skew-symmetric matrices is called its **Lie algebra**.

3. Consider a vector $\omega \in \mathbb{R}^3$ with $\|\omega\| = 1$ and its corresponding skew-symmetric matrix $\hat{\omega}$.
   (a) Show that $\hat{\omega}^2 = \omega\omega^\top - I$ and $\hat{\omega}^3 = -\hat{\omega}$.
   (b) Following the result of (a), find simple rules for the calculation of $\hat{\omega}^n$ and proof your result. Distinguish between odd and even numbers n.
   (c) Derive the Rodrigues' formula for a skew-symmetric matrix $\hat{\omega}$ corresponding to an arbitrary vector $\omega \in \mathbb{R}^3$ (i.e. $\|\omega\|$ does not have to be equal to 1):

$$e^{\hat{\omega}} = I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|))$$

   Hint: Combine your result from (b) with

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} \quad \text{and} \quad \sin(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \quad \text{and} \quad 1-\cos(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}$$