# Multiple View Geometry: Solution Sheet 1

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1. State the definition of a group
   A group is a tuple of
   - set $M$
   - operation $\cdot : M \times M \to M, (a, b) \mapsto a \cdot b$

   with the properties
   - **associative:** $\forall a, b \in M : (a \cdot b) \cdot c = a \cdot (b \cdot c)$
   - **identity:** $\exists e \in M : \forall a \in M : a \cdot e = a$
   - **inverse:** $\forall a \in M : \exists a^{-1} \in M : a \cdot a^{-1} = e$

2. Let $(M, \cdot)$ be a group, with the right identity $e \in M$, $a \cdot e = a$, and the right inverse element $a^{-1} \in M$, $a \cdot a^{-1} = e$. Show that and the right identity is also the left identity and the right inverse is also the left inverse element.

   Left identity: Let, $f \cdot a = a$. Use definition of right identity as well. $f \cdot e = e = f$

   Left inverse: Let, $b \cdot a = e$. $b = b \cdot e = b \cdot a \cdot a^{-1} = ea^{-1} = a^{-1}$

3. Let $(M, \cdot)$ be a group, show that the inverse element $a^{-1} \in M$ of $a \in M$ is unique Assume there exists a $c \in M$ with $ac = e$.
   $c = c \cdot e = c(aa^{-1}) = (ca)a^{-1} = a^{-1}$

4. Is the following statement correct? For groups, whose operation does not fulfil the commutative property (e.g. matrix multiplication) the left and a right inverse elements are distinct.

   Not correct, as shown in previous exercise. We showed it for all groups, so it also holds for abelsch (commutative) groups.

5. Which of the following sets forms a group (with matrix-multiplication)? Prove or disprove!
   Definition of a subgroup:
   A subset $G_i \subset G$ is a subgroup of a group, if it also forms a group under the same operation $\cdot$.
   **Subgroup tests:**
   - closed under operation: $\forall A, B \in G_i : A \cdot B \in G_i$
   - existence of inverse element: $\forall A \in G_i : A^{-1} \in G_i$

   This implies, that the identity of $G$ is also an element of $G_i$. However, if the identity is not an element of $G_i$ we can quickly show, that $G_i$ is not a group.

   All sets are subsets of the set of invertible matrices $GL(n) = \{A \in \mathbb{R}^{n\times n} | \det(A) \neq 0\}$.

   (a) $G_1 := \left\{A \in \mathbb{R}^{n\times n} | \det(A) \neq 0 \wedge A^\top = A\right\}$
   Set of symmetric invertible matrices.

- $\forall A, B \in G_1$, the multiplication $AB$ has to be an element of $G_1$. But in general $(AB)^\top = B^\top A^\top = BA \neq AB$, as the matrix multiplication is not commutative.

(b) $G_2 := \{A \in \mathbb{R}^{n\times n} | \det(A) = -1\}$

- $\det(I) = 1 \neq -1 \Rightarrow I \notin G_2$ The set contains no neutral element, thus it is not a group.
- $\forall A, B \in G_2$, the multiplication $AB$ has to be an element of $G_2$. But in general $\det(AB) = \det(A)\det(B) = -1 \cdot -1 = 1 \neq -1$

(c) $G_3 := \{A \in \mathbb{R}^{n\times n} | \det(A) > 0\}$
Recall the definition of subgroup

- $I \in G_3$
- $\forall A \in G_3$, the inverse $A^{-1}$ has to be an element of $G_3$.
$\det(A^{-1}) = \frac{1}{\underbrace{\det(A)}_{>0}} > 0$
- $\forall A, B \in G_3$, the multiplication $AB$ has to be an element of $G_3$.
$\det(AB) = \underbrace{\det(A)}_{>0}\underbrace{\det(B)}_{>0} > 0$

Thus, $G_3$ is a subgroup of $GL(n)$ and hence a group.

6. Groups and inclusions:
Groups

(a) $SO(n)$: special orthogonal group
(b) $O(n)$: orthogonal group
(c) $GL(n)$: general linear group
(d) $SL(n)$: special linear group
(e) $SE(n)$: special euclidean group (In particular, $SE(3)$ represents the rigid-body motions in $\mathbb{R}^3$)
(f) $E(n)$: euclidean group
(g) $A(n)$: affine group

Inclusions

(a) $SO(n) \subset O(n) \subset GL(n)$
(b) $SE(n) \subset E(n) \subset A(n) \subset GL(n+1)$

7. State the definition of a vector space $V$ over a field $\mathbb{K}$(which is eiher $\mathbb{C}$ or $\mathbb{K}$). Neglect the definition of a field here. Does $V$ have to fulfil the group properties? What additional properties does a vectorspace fulfil?

A set $V$ with operation $+$ is over a field $\mathbb{K}$ is a vector space if we have

- $(V, +)$ is a commutative group
- scalar multiplication $\cdot : \mathbb{K} \times V \to V, \lambda \cdot A \mapsto B$
  - $1\mathbf{v} = \mathbf{v}$ (identity elem. of scalar mul.)
  - $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$ (compatibility of scalar mul. with field mul.)

- $\alpha(\mathbf{v}+\mathbf{w}) = \alpha\mathbf{v}+\beta\mathbf{w}$, $(\alpha+\beta)\mathbf{v} = \alpha\mathbf{v}+\beta\mathbf{v}$ (distributivity of scalar mul. wrt. vector add.)

- field is set $\mathbb{K}$ which forms commutative groups $(\mathbb{K}, +)$, $(\mathbb{K} \setminus \{0\}, \cdot)$, and fulfils the distibutive property

8. Let $V$ be a vector space over $\mathbb{K}$. State the definition of

   - linear independence of pairwise distinct $\mathbf{v}_1, ..., \mathbf{v}_k \in V$
     $\sum_{i=1}^{k} \alpha_i \mathbf{v}_i = 0 \Rightarrow \alpha_i = 0 \forall \alpha_i$
   - the span of a set $M \subset V$
     $\text{span}(M) = \left\{ \sum_{i=1}^{k} \alpha_i \mathbf{v}_i \;\middle|\; \mathbf{v}_i \in M, \alpha_i \in \mathbb{K} \right\}$
   - the basis of $U \subset V$.
     linearly independent set $M$ that spans $U$. That is, $U = \text{span}(M)$ with lin. indep. M.

9. Show (without using concepts like determinant) for each of the following sets (1) whether they are linearly independent, (2) whether they span $\mathbb{R}^3$ and (3) whether they form a basis of $\mathbb{R}^3$:

   (a) $M_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

   - is linearly independent, as $\sum_i \alpha_i \mathbf{v}_i = 0 \Rightarrow \alpha_1 = 0 \Rightarrow \alpha_2 = 0 \Rightarrow \alpha_3 = 0$
   - spans $\mathbb{R}^3$, as for any $x = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]^T$ we have $\mathbf{x} = x_3\mathbf{v}_3 + x_2(\mathbf{v}_2 - \mathbf{v}_3) + x_1(\mathbf{v}_1 - \mathbf{v}_2)$ and thus $\mathbf{x} \in \text{span}(M_1)$
   - forms a basis of $\mathbb{R}^3$, as its elements are independent and span $\mathbb{R}^3$.

   (b) $M_2 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

   - is linearly independent, as $\sum_i \alpha_i \mathbf{v}_i = 0 \Rightarrow \alpha_1 = 0 \Rightarrow \alpha_2 = 0$
   - does not span $\mathbb{R}^3$, as for any $x = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]^T$ with $x_3 \neq 0$ we cannot find a lin. comb.
   - does not form a basis of $\mathbb{R}^3$, as it does not span $\mathbb{R}^3$

   (c) $M_3 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

   - is linearly dependent, as $\sum_i \alpha_i \mathbf{v}_i = 0 \Rightarrow [\alpha_1, \alpha_2, \alpha_3, \alpha_4] = \lambda[1, -1, -1, 1] \neq 0$
   - spans $\mathbb{R}^3$, as for any $x = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]^T$ we have $\mathbf{x} = x_3(\mathbf{v}_4 - \mathbf{v}_3) + x_1(\mathbf{v}_2 - \mathbf{v}_1) + x_2(\mathbf{v}_1 - 2(\mathbf{v}_2 - \mathbf{v}_1))$ and thus $\mathbf{x} \in \text{span}(M_3)$
   - does not form a basis of $\mathbb{R}^3$, as vectors are lin. dependent

10. The dimension theorem for vector spaces states: *Given a vector space $V$, any two bases have the same cardinality. This number defines the dimension of the vector space.*
    Show by using the previous exercise: In $\mathbb{R}^3$, there cannot be more than three independent vectors.
    We have found a basis of cardinality three in the previous exercise. As there can not be found a basis with different cardinality, we cannot find 4 lin.indep. vec. that span $\mathbb{R}^3$, as this would fulfil the definition of a basis. But the theorem only allows a basis with three lin. indep. vectors.

11. A hilbert space $H$ is a finite dimensional vector space over a field $\mathbb{K}$ endowed with an inner product. State the definition of an inner product.

A function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$

- Symmetry: $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{u} \rangle$
- Linear in the second argument: $\langle \mathbf{v}, \alpha\mathbf{u} \rangle = \alpha\langle \mathbf{v}, \mathbf{u} \rangle$, $\langle \mathbf{v}, \mathbf{u} + \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
- Positive definiteness: if $\mathbf{v}$ is not zero, then $\langle \mathbf{v}, \mathbf{v} \rangle > 0$

12. State for the following, whether the following Vector spaces form a Hilbert space with the provided inner product.

- $\mathbb{R}^n$ with $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y}$
  - Symm: $\mathbf{x}^T\mathbf{y} = x_1y_1 + ... = y_1x_1 + ... = \mathbf{y}^T\mathbf{x}$
  - Lin: $\mathbf{x}^T(\alpha\mathbf{y}) = x_1\alpha y_1 + ... = \alpha x_1y_1 + ... = (\alpha\mathbf{x})^T\mathbf{y}$
    $\mathbf{x}^T(\mathbf{y} + \mathbf{z}) = x_1(y_1 + z_1) + ... = x_1y_1 + x_1z_1 + ... = \mathbf{x}^T\mathbf{y} + \mathbf{x}^T\mathbf{z}$
  - Pos. def: $\mathbf{x}^T\mathbf{x} = x_1^2 + ... = \sum_i x_i^2 \geq 0$, now $\mathbf{x} \neq 0 \Rightarrow$ some $x_i \neq 0$ and thus $\sum_i x_i^2 > 0$
- $\mathbb{R}^{n\times m}$ with $\langle A, B \rangle = \mathrm{tr}(A^TB)$
  $A = [\mathbf{a}_1, ..., \mathbf{a}_n]$, $B = [\mathbf{b}_1, ..., \mathbf{b}_n]$
  - Symm: $\mathrm{tr}(A^TB) = \sum_k \mathbf{a}_k^T\mathbf{b}_k = \sum \mathbf{b}_k^T\mathbf{a}_k = \mathrm{tr}(B^TA)$ with using the previous subproblem
  - Lin: $\mathrm{tr}(A^T\alpha B) = \sum_k \mathbf{a}_k^T\alpha\mathbf{b}_k = \sum \alpha\mathbf{b}_k^T\mathbf{a}_k = \mathrm{tr}(\alpha B^TA)$ with using the previous subproblem
    $\mathrm{tr}(A^T(B+C)) = \sum_k \mathbf{a}_k^T(\mathbf{b}_k + \mathbf{c}_k) = \sum \mathbf{a}_k^T\mathbf{b}_k \sum \mathbf{a}_k^T\mathbf{c}_k = \mathrm{tr}(\alpha A^TB) + \mathrm{tr}(\alpha A^TC)$ with using the previous subproblem
  - Pos. def.: Let $\mathrm{tr}(A^TA) = \sum_k \mathbf{a}_k^T\mathbf{a}_k = 0$. As $\mathbf{a}_k^T\mathbf{a}_k \geq 0$ from above holds in general we can infer $\mathbf{a}_k^T\mathbf{a}_k = 0 \forall k$ and further $\mathbf{a}_k = 0$ by using the previous exercise and thus $A = 0$

13. Prove or disprove: There exist vectors $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, which are pairwise orthogonal, i.e.

$$\forall i, j = 1, ..., 5 : \quad i \neq j \implies \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$$

Hint: From the previous problem you can use: *In $\mathbb{R}^3$, there are at most three linearly independent vectors.* Assume there exist five pairwise orthogonal, non-zero vectors $\mathbf{v}_1, ..., \mathbf{v}_5 \in \mathbb{R}^3$. In $\mathbb{R}^3$, there are at most three linearly independent vectors. Thus, the vectors are linearly dependent, which means

$$\exists\, a_i : \quad \sum_{i=1}^{5} a_i\mathbf{v}_i = 0\,,$$

with at least one $a_i \neq 0$. Without loss of generality, assume that $a_1 = -1$, resulting in

$$\mathbf{v}_1 = a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5$$

As the vectors are assumed to be pairwise orthogonal, we can derive

$$\begin{aligned}\langle \mathbf{v}_1, \mathbf{v}_1 \rangle &= \langle \mathbf{v}_1, a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 + a_5\mathbf{v}_5 \rangle = \\ &= a_2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle + a_3\langle \mathbf{v}_1, \mathbf{v}_3 \rangle + a_4\langle \mathbf{v}_1, \mathbf{v}_4 \rangle + a_5\langle \mathbf{v}_1, \mathbf{v}_5 \rangle = \\ &= 0 + 0 + 0 + 0 = 0 \\ \Rightarrow \quad \mathbf{v}_1 &= \mathbf{0}\,,\end{aligned}$$

which contradicts the assumption of pairwise orthogonal, non-zero vectors.

14. Show that the frobenius norm $\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}^2}$ for $A \in \mathbb{R}^{n\times m}$ is an induced norm of the inner product $\langle A, B\rangle = \operatorname{tr}(A^\top B)$.

    $\operatorname{tr}(A^\top A) = \sum_i a_i^T a_i = \sum_i \sum_j a_{ij}^2$

    $\Rightarrow \|A\|_F = \sqrt{\sum_i \sum_j a_{ij}^2} = \sqrt{\langle A, A\rangle}$