Multiple View Geometry: Solution Sheet 2

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`Wednesdays 16:15-17:45 at Hörsaal 2, "Interims I" (5620.01.102), and on RBG Live`

Exercise: May 8th, 2024

Part I: Theory

1. Rigid body motion requires to preserve 1) the norm and 2) the cross product. We first show the norm preservation, given a vector $v \in \mathbb{R}^3$ and a rotation matrix $R \in \mathbb{R}^{3\times 3}$:

$$||Rv||^2 = (Rv)^\top Rv = v^\top R^\top Rv = v^\top v = ||v||^2$$

Next, we show the cross product preservation. Given two vectors a and b and assume we only rotate around x-axis with angle θ. Therefore, the rotation matrix is:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Rotating the cross product of a and b with $R_x(\theta)$, we have:

$$\begin{aligned} a \times b &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \\ \Rightarrow R_x(\theta)(a \times b) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \\ &= \begin{pmatrix} a_2b_3 - a_3b_2 \\ \cos(\theta)(a_3b_1 - a_1b_3) - \sin(\theta)(a_1b_2 - a_2b_1) \\ \sin(\theta)(a_3b_1 - a_1b_3) + \cos(\theta)(a_1b_2 - a_2b_1) \end{pmatrix} \end{aligned}$$

Now, we first rotate a and b with $R_x(\theta)$ and then calculate the cross product:

$$\begin{aligned} (R_x(\theta)a) \times (R_x(\theta)b) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ \cos(\theta)a_2 - \sin(\theta)a_3 \\ \sin(\theta)a_2 + \cos(\theta)a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ \cos(\theta)b_2 - \sin(\theta)b_3 \\ \sin(\theta)b_2 + \cos(\theta)b_3 \end{pmatrix} \end{aligned}$$

Following the caculation of the cross product, we can see that the two results are the same. Therefore, the cross product is preserved under rotation around the x-axis. The same can be shown for the y and z axis. The corresponding rotation matrices are:

$$R_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \qquad R_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. $\lambda_a = \frac{(\lambda_a v_a)^\top v_b}{\langle v_a, v_b\rangle} = \frac{v_a^\top A^\top v_b}{\langle v_a, v_b\rangle} = \frac{v_a^\top A v_b}{\langle v_a, v_b\rangle} = \frac{v_a^\top (\lambda_b v_b)}{\langle v_a, v_b\rangle} = \lambda_b$

3. We assume $u \neq -v$. Otherwise, we only get $w = 0$. We first compute the inner product between u and v:

$$\langle u, v\rangle = \|u\|\|v\|\cos(\theta) = 1 \cdot 1 \cdot \cos(\theta) = \cos(\theta)$$

Next, we compute the inner product between w and u:

$$\langle w, u\rangle = \langle u + v, u\rangle = \langle u, u\rangle + \langle v, u\rangle = 1 + \cos(\theta)$$

Alternatively, we can compute the inner product between w and u as follows:

$$\langle w, u\rangle = \|w\|\|u\|\cos(\alpha) = \|u + v\|\cos(\alpha)$$

Now, we compute $\|u + v\|$:

$$\|u + v\| = \sqrt{\langle u + v, u + v\rangle} = \sqrt{\langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle} = \sqrt{1 + 2\cos(\theta) + 1} = \sqrt{2(1 + \cos(\theta))}$$

Finally, we compute $\cos(\alpha)$:

$$\begin{aligned} \|u + v\|\cos(\alpha) &= 1 + \cos(\theta) \\ \Rightarrow \sqrt{2(1 + \cos(\theta))}\cos(\alpha) &= 1 + \cos(\theta) \\ \Rightarrow \cos(\alpha) &= \frac{1 + \cos(\theta)}{\sqrt{2(1 + \cos(\theta))}}. \end{aligned}$$

Because $\cos^2(\theta/2) = \frac{1+\cos(\theta)}{2}$, we have shown $\alpha = \theta/2$.

4. Let V be the orthonormal matrix (i.e. $V^\top = V^{-1}$) given by the eigenvectors, and Σ the diagonal matrix containing the eigenvalues:

$$V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \lambda_1 & 0 & \ddots \\ 0 & \ddots & 0 \\ \ddots & 0 & \lambda_n \end{pmatrix}.$$

As V is a basis, we can express x as a linear combination of the eigenvectors $x = V\alpha$, for some $\alpha \in \mathbb{R}^n$. For $||x|| = 1$ we have $\sum_i \alpha_i^2 = \alpha^\top\alpha = x^\top V V^\top x = x^\top x = 1$. This gives

$$\begin{aligned} x^\top A x &= x^\top V \Sigma V^{-1} x \\ &= \alpha^\top V^\top V \Sigma V^\top V \alpha \\ &= \alpha^\top \Sigma \alpha = \sum_i \alpha_i^2 \lambda_i \end{aligned}$$

Considering $\sum_i \alpha_i^2 = 1$, we can conclude that this expression is minimized iff only the α_i corresponding to the smallest eigenvalue(s) are non-zero. If $\lambda_{n-1} \gneq \lambda_n$, there exist only two solutions ($\alpha_n = \pm 1$), otherwise infinitely many.

For maximisation, only the the α_i corresponding to the largest eigenvalue(s) can be non-zero.

5. We show that: $\quad x \in \text{kernel}(A) \Leftrightarrow x \in \text{kernel}(A^\top A)$.

   "$\Rightarrow$": Let $x \in \text{kernel}(A)$

   $$A^\top \underbrace{Ax}_{=0} = A^\top 0 = 0 \quad \Rightarrow x \in \text{kernel}(A^\top A)$$

   "$\Leftarrow$": Let $x \in \text{kernel}(A^\top A)$

   $$0 = x^\top \underbrace{A^\top Ax}_{=0} = \langle Ax, Ax\rangle = ||Ax||^2 \quad \Rightarrow Ax = 0 \quad \Rightarrow x \in \text{kernel}(A)$$

6. Singular Value Decomposition (SVD)
   Note: There exist multiple slightly different definitions of the SVD. Depending on the convention used, we might have $S \in \mathbb{R}^{m\times n}$, $S \in \mathbb{R}^{n\times n}$, or $S \in \mathbb{R}^{p\times p}$ where $p = \text{rank}(A)$. In the lecture the first option was presented. In the following, we present the results for the same option, since that is the one that `numpy.linalg.svd` function returns by default.

   (a) $A \in \mathbb{R}^{m\times n}, U \in \mathbb{R}^{m\times m}, S \in \mathbb{R}^{m\times n}, V \in \mathbb{R}^{n\times n}$

   (b) Similarities and differences between SVD and EVD:

      i. Both are matrix diagonalization techniques.

      ii. The SVD can be applied to matrices $A \in \mathbb{R}^{m\times n}$ with $m \neq n$, whereas the EVD is only applicable to quadratic matrices ($A \in \mathbb{R}^{m\times n}$ with $m = n$).

   (c) Relationship between U, S, V and the eigenvalues and eigenvectors of $A^\top A$ and $AA^\top$:

      i. $A^\top A$: The columns of V are eigenvectors; the squares of the diagonal elements of S are eigenvalues.

      ii. $AA^\top$: The columns of U are eigenvectors; the squares of the diagonal elements of S are eigenvalues (possibly filled up with zeros).

   (d) Entries in S:

      i. S is a diagonal matrix. The elements along the diagonal are the *singular values* of A.

      ii. The number of non-zero singular values gives us the *rank* of the matrix A.