Multiple View Geometry: Solution Sheet 3

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Part I: Theory

1. (a) $M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (b) $M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (c) $M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (d) $M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & RT \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_1 T \\ r_{21} & r_{22} & r_{23} & r_2 T \\ r_{31} & r_{32} & r_{33} & r_3 T \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

   where r_1, r_2, r_3 are the row vectors of R: $R = \begin{pmatrix} — r_1 — \\ — r_2 — \\ — r_3 — \end{pmatrix}$.

2. Let $M := (M_1 - M_2) =: \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$.

   ”⇒”:
   We show that M is skew-symmetric by distinguishing diagonal and off-diagonal elements of M:

   (a) $\forall i\colon 0 = e_i^\top M e_i = m_{ii}$ where $e_i =$ i-th unit vector

   (b) $\forall i \neq j\colon 0 = (e_i + e_j)^\top M (e_i + e_j)$ where $e_j =$ j-th unit vector
   $= m_{ii} + m_{jj} + m_{ij} + m_{ji} \Rightarrow m_{ij} = -m_{ji}$

   hence, $m_{ii} = 0$ and $m_{ij} = -m_{ji}$, i.e. M is skew-symmetric.

"$\Leftarrow$":
using $M = -M^\top$, we directly calculate

$$\begin{aligned} \forall x\colon x^\top M x &= (x^\top M x)^\top = x^\top M^\top x = -(x^\top M x) \\ &\Rightarrow x^\top M x = 0 \end{aligned}$$

Alternative for "$\Leftarrow$":

$$\forall x\colon x^\top M x = x^\top (\check{M} \times x) = 0$$

Because M is skew-symmetric, Mx can be interpreted as a cross product. The result of any cross product with x is orthogonal to x.

3. We know: $\omega = (\omega_1\ \omega_2\ \omega_3)^\top$ with $||\omega|| = 1$ and $\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$.

   (a)

$$\begin{aligned} \hat{\omega}^2 &= \begin{pmatrix} -(\omega_2^2+\omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2+\omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2+\omega_2^2) \end{pmatrix} \\ &= \begin{pmatrix} \omega_1^2 - \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 - \underbrace{(\omega_2^2+\omega_1^2+\omega_3^2)}_{1} & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 - \underbrace{(\omega_3^2+\omega_1^2+\omega_2^2)}_{1} \end{pmatrix} \\ &= \begin{pmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \omega\omega^\top - \mathrm{I} \end{aligned}$$

$$\begin{aligned} \hat{\omega}^3 &= \hat{\omega}\,\hat{\omega}^2 \\ &= \hat{\omega}\,(\omega\omega^\top - I) \\ &= \hat{\omega}\,\omega\,(\omega^\top) - \hat{\omega} I \\ &= (\omega \times \omega)\,\omega^\top - \hat{\omega} \\ &= -\hat{\omega} \qquad \text{(as } \omega \times \omega = 0\text{)} \end{aligned}$$

Alternative solution for $\hat{\omega}^3$:

$$
\begin{aligned}
\hat{\omega}^3 &= \begin{pmatrix} -(\omega_2^2+\omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2+\omega_2^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2+\omega_2^2) \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \omega_3 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & -\omega_2 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} \\ -\omega_3 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & 0 & \omega_1 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} \\ \omega_2 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & -\omega_1 \cdot \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & 0 \end{pmatrix} \\
&= -\hat{\omega}
\end{aligned}
$$

(b) The formulas for n even and odd can be found by writing down the solutions for $n = 1, \ldots, 6$:

$$
\begin{array}{lcll}
\hat{\omega} & & & \\
\hat{\omega}^2 & & & \\
\hat{\omega}^3 & = & -\hat{\omega} & \\
\hat{\omega}^4 & = & -\hat{\omega}^2 & \text{as: } \hat{\omega}^4 = \hat{\omega}^3\hat{\omega} = -\hat{\omega}\hat{\omega} = -\hat{\omega}^2 \\
\hat{\omega}^5 & = & \hat{\omega} & \text{as: } \hat{\omega}^5 = \hat{\omega}^4\hat{\omega} = -\hat{\omega}^2\hat{\omega} = -\hat{\omega}^3 = -(-\hat{\omega}) = \hat{\omega} \\
\hat{\omega}^6 & = & \hat{\omega}^2 & \text{as: } \hat{\omega}^6 = \hat{\omega}^5\hat{\omega} = \hat{\omega}\hat{\omega} = \hat{\omega}^2
\end{array}
$$

For even numbers:

$$
\begin{array}{lcl}
\hat{\omega}^2 & & \\
\hat{\omega}^4 & = & -\hat{\omega}^2 \\
\hat{\omega}^6 & = & \hat{\omega}^2
\end{array}
$$

For odd numbers:

$$
\begin{array}{lcl}
\hat{\omega} & & \\
\hat{\omega}^3 & = & -\hat{\omega} \\
\hat{\omega}^5 & = & \hat{\omega}
\end{array}
$$

$$
\begin{array}{llcll}
\text{even:} & \hat{\omega}^{2n} & = & (-1)^{n+1}\,\hat{\omega}^2 & \text{for } n \geq 1 \\
\text{odd:} & \hat{\omega}^{2n+1} & = & (-1)^{n}\,\hat{\omega} & \text{for } n \geq 0
\end{array}
$$

Proof via complete induction:

i. For even numbers $2n$ where $n \geq 1$:
   - $n = 1:\quad \hat{\omega}^2 = (-1)^2\hat{\omega}^2$
   - Induction step $n \to n+1$:

$$
\begin{aligned}
\hat{\omega}^{2(n+1)} &= \hat{\omega}^{2n} \cdot \hat{\omega}^2 \\
&= (-1)^{n+1} \cdot \hat{\omega}^2 \cdot \hat{\omega}^2 \qquad \text{(assumption)} \\
&= (-1)^{n+1} \cdot \hat{\omega}^3 \cdot \hat{\omega} \\
&\overset{(a)}{=} (-1)^{(n+1)+1} \cdot \hat{\omega}^2
\end{aligned}
$$

ii. For odd numbers $2n+1$ where $n \geq 0$:

- $n = 0:\quad \hat{\omega}^1 = (-1)^0 \hat{\omega}$
- Induction step $n \to n+1$:

$$\begin{aligned}
\hat{\omega}^{2(n+1)+1} &= \hat{\omega}^{2n+1} \cdot \hat{\omega}^2 \\
&= (-1)^n \cdot \hat{\omega} \cdot \hat{\omega}^2 && \text{(assumption)} \\
&= (-1)^n \cdot \hat{\omega}^3 \\
&\overset{(a)}{=} (-1)^{n+1} \cdot \hat{\omega}
\end{aligned}$$

(c) We know: $\omega \in \mathbb{R}^3$. Let $\nu = \frac{\omega}{\|\omega\|}$ and $t = \|\omega\|$. Hence, $w = \nu t$, $\hat{\omega} = \hat{\nu} t$.

$$\begin{aligned}
e^{\hat{\omega}} &= e^{\hat{\nu} t} \\
&= \sum_{n=0}^{\infty} \frac{(\hat{\nu} t)^n}{n!} \\
&= I + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\, \hat{\nu}^{2n} + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\, \hat{\nu}^{2n+1} \\
&\overset{(b)}{=} I + \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}}_{1-\cos(t)}\, \hat{\nu}^2 + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}}_{\sin(t)}\, \hat{\nu} \\
&\overset{\text{(def.)}}{=} I + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|)) + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|)
\end{aligned}$$