Combinatorial Optimization in Computer Vision

Chapter 12: Submodularity and Quadratic Pseudo-Boolean Functions

WS 2011/12

Ulrich Schlickewei
Computer Vision and Pattern Recognition Group
Technische Universität München

Plan

In this lecture we will study the notion of submodularity in the important special case of quadratic pseudo-Boolean functions.

In particular, we will see

- an easy criterion for checking whether a given quadratic pseudo-Boolean function is submodular
- that submodular quadratic pseudo-Boolean functions can be minimized in polynomial time by computing a minimal cut uin an apprmedian graph.

Quadratic Pseudo-Boolean Functions

Definition: Let $V = \{1, \dots, n\}$ and let $f : \{0,1\}^V \to \mathbb{R}^n$ be a pseudo-Boolean function. Then f is quadratic if the corresponding multilinear polynomial has degree two, that is

$$f(x_1, \dots, x_n) = c_0 + \sum_{i=1}^{n} c_i x_i + \sum_{1 \le i < j \le n} c_{ij} x_i x_j .$$

Remarks:

- A quadratic pseudo-Boolean can also be represented by a posiform of degree two.

Remarks

- There are posiforms of degree larger than two which represent quadratic pseudo-Boolean functions. For example, the function

$$f(x_1, \ldots, x_n) = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_2x_3 + x_1x_3$$

  can be represented by the cubic posiform

$$\phi(x_1, x_2, x_3) = x_1x_2x_3 + \overline{x_1x_2x_3}.$$

- We have seen in Chapter 10 that any pseudo-Boolean optimization problem can be reduced to a quadratic pseudo-Boolean optimization problem.

Submodularity

- In the last chapter, we introduced the concept of submodular functions.
- We have seen tht submodular pseudo-Boolean optimization can be reduced to continuous convex optimization by means of the Lovasz extension.
- Moreover, by the theorem of Grötschel, Lovasz and Schrijver, submodular optimization problems can be solved in polynomial time.

Characterization of Submodular Quadratic Pseudo-Boolean Functions

Prop.: Let $f : \{0,1\}^V \to \mathbb{R}^n$ be a quadratic pseudo-Boolean function. Then the following are equivalent.

a) f is submodular.

b) In the multilinear polynomial representation of f

$$f(x_1, \ldots, x_n) = c_0 + \sum_{i=1}^{n} c_i x_i + \sum_{1 \leq i < j \leq n} c_{ij} x_i x_j.$$

the quadratic cofficients are non-positive, that is $c_{ij} \leq 0.$

c) There exists a partition $V = W_0 \cup W_1$ and a posiform representation of f

$$\phi(x_1, \ldots, x_n) = a_0 + \sum_{i \in W_0} a_{\bar{i}} \overline{x_i} + \sum_{i \in W_1} a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i \overline{x_j}.$$

Graph Representation of Submodular Quadratic Pseudo-Boolean Functions I

Let $f : \{0,1\}^V \to \mathbb{R}^n$ be a submodular quadratic function. Consider a partition $V = W_0 \cup W_1$ and a posiform representation

$$\phi(x_1, \ldots, x_n) = a_0 + \sum_{i \in W_0} a_{\bar{i}} \overline{x_i} + \sum_{i \in W_1} a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i \overline{x_j}.$$

We define a network $N = (W, E, s, t, c)$ with

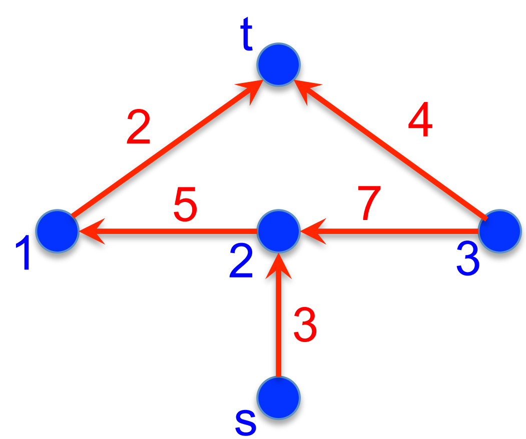
- $W = V \cup \{s_0, t_1\}.$
- $s = s_0,\ t = t_1$
- an edge $e_{s_0,i}$ connecting s_0 with each vertex $i \in W_1$, $c(e_{s_0,i}) = a_i$
- an edge e_{i,t_1} connecting each vertex $i \in W_0$, $c(e_{s_0,i}) = a_{\bar{i}}$

Graph Representation of Submodular Quadratic Pseudo-Boolean Functions II

- an edge e_{ji} connecting vertex j with vertex i if $j > i$ and if $a_{ij} > 0$, capacity $c(e_{ji}) = a_{ij}$

Example: Consider the posiform

$$\phi(x_1, x_2, x_3) = 2\overline{x_1} + 4\overline{x_3} + 3x_2 + 5x_1\overline{x_2} + 7x_2\overline{x_3}$$



Minimization by Graph Cuts

Prop.: Let $C = S \cup T$ be a cut in the network $N = (W, E, s, t, c)$.
Let $x_C \in \{0,1\}^V$ be defined as

$$x_C(i) \begin{cases} 0 & \text{if } i \in S \\ 1 & \text{if } i \in T. \end{cases}$$

Then the cost of C coincides with $f(x_C)$ up to the constant term a_0 of the posiform ϕ, that is $|C| + a_0 = f(x_C)$.

Since all weights are positive, we can compute a min-cut in the network in polynomial time. This gives a proof of the Grötschel-Lovasz-Schrijver theorem in the quadratic case.

Summary

- Submodular quadratic pseudo-Boolean problems are characterized by a multilinear polynomial representation with non-positive quadratic terms.
- Submodular quadratic pseudo-Boolean functions can be minimized by computing a min-cut in a network with non-negative edge weights.
- The quadratic pseudo-Boolean MRF-inference problems we have seen before (binary segmentation, computing the optimal alpha-expansion) are examples of submodular quadratic pseudo-Boolean optimization problems.