Combinatorial Optimization in Computer Vision

Chapter 13: Roof Duality

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Minimization of General Quadratic Pseudo-Boolean Functions

- In the last chapter, we have seen, that submodular quadratic pseudo-Boolean functions can be minimized globally in polynomial time by computing a min-cut in an appropriate network.

- In this chapter, we will tackle the problem of minimizing general quadratic pseudo-Boolean functions approximately. We will see an algorithm, in the vision community known as QPBO, which
  - gives a lower bound on the optimal value
  - computes a partial assignment at which persistency holds.

Submodular Relaxation

Let $f : \{0,1\}^n \to \mathbb{R}$ with $f(0) = 0$. Instead of considering directly the optimization problem

$$\min_{x \in \{0,1\}^n} f(x)$$

we will construct a submodular quadratic function

$$g : \{0,1\}^{2n} \to \mathbb{R}$$

satisfying

$$g(x, \overline{x}) = f(x) \quad \forall x \in \{0,1\}^n.$$

Lower Bound

Then clearly, we obtain a lower bound

$$\min_{(x,y)\in\{0,1\}^{2n}} g(x,y) \leq \min_{x\in\{0,1\}^n} f(x).$$

Furthermore, given

$$(x^*, y^*) \in \operatorname{argmin}_{(x,y)\in\{0,1\}^{2n}} g(x,y)$$

we can construct a partial labelling at those indices for which

$$y_i = \overline{x_i} = (1 - x_i).$$

(Remember that $g(x, \overline{x}) = f(x) \quad \forall x \in \{0,1\}^n.$)

Best Lower Bound

In order to construct the best approximation possible, we will solve (explicitely) the maximization part of the following optimization problem:

$$
\begin{aligned}
\max_{g:\{0,1\}^{2n}\to\mathbb{R}} \min_{x,y\in\{0,1\}^n} \quad & g(x,y) \\
\text{subject to} \quad & g \text{ submodular} \\
& g(x,\overline{x}) = f(x) \quad \forall x \in \{0,1\}^n.
\end{aligned}
$$

Plan

We will proceed as follows:

- Notation
- Symmetry
- Bisubmodularity
- Explicit solution of the maximization problem
- Persistency

Notation I

- For $x \in \{0,1\}^n$: $\overline{x} = (1 - x_1, \ldots, 1 - x_n)$.

- For $x, y \in \{0,1\}^n$

$$x \wedge y = \Big(\min(x_1, y_1), \ldots, \min(x_n, y_n)\Big)$$

$$x \vee y = \Big(\max(x_1, y_1), \ldots, \max(x_n, y_n)\Big)$$

Note that if $x = \mathbf{1}_S, y = \mathbf{1}_T$, then $x \wedge y = \mathbf{1}_{S \cap T}$ and $x \wedge y = \mathbf{1}_{S \cup T}$.

- A pseudo-Boolean function h is submodular if and only if $h(x) + h(y) \geq h(x \wedge y) + h(x \vee y)$.

Notation II

- $S^n = \{(x,y) \in \{0,1\}^{2n} \mid \forall i \in \{1,\ldots,n\} : \ (x_i, y_i) \neq (1,1)\}.$

- For $(x_1,y_1),(x_2,y_2) \in S^n :$

$$(x_1,y_1) \sqcap (x_2,y_2) := (x_1 \wedge x_2, y_1 \wedge y_2),$$

$$(x_1,y_1) \sqcup (x_2,y_2) := ((x_1 \vee x_2) \wedge \overline{(y_1 \vee y_2)}, (y_1 \vee y_2) \wedge \overline{(x_1 \vee x_2)}).$$

Symmetry

Let $g : \{0,1\}^{2n} \to \mathbb{R}$, then g is symmetric if

$$g(x,y) = g(\overline{y}, \overline{x}) \quad \forall (x,y) \in \{0,1\}^{2n}.$$

It turns out, that in order to solve

$$\begin{aligned} \max_{g:\{0,1\}^{2n}\to\mathbb{R}} \min_{x,y\in\{0,1\}^n} \quad & g(x,y) \\ \text{subject to} \quad & g \text{ submodular} \\ & g(x,\overline{x}) = f(x) \quad \forall x \in \{0,1\}^n. \end{aligned}$$

it is enough to optimize over symmetric functions g.

Bisubmodularity

Lemma: Let $g : \{0,1\}^{2n} \to \mathbb{R}$ be symmetric and submodular.

a) Let $(x,y) \in \{0,1\}^{2n}$. Then

$$g(x,y) \geq g(x \wedge \overline{y}, y \wedge \overline{x})$$

Thus, when minimizing g, it is enough to focus on vectors in S^n.

b) g is bisubmodular, that is

$$g(x_1,y_1) + g(x_2,y_2) \geq g((x_1,y_1) \sqcup (x_2,y_2)) + g(x_1,y_1) \sqcap (x_2,y_2)).$$

for all $(x_1,x_2),(y_1,y_2) \in \{0,1\}^{2n}$.

Explicit Solution of the max-min Problem

Assume that $f(x) = \sum_i c_i x_i + \sum_{i<j} c_{ij} x_i x_j.$

Then the symmetric, submodular function $g : \{0,1\}^{2n} \to \mathbb{R}$ given by

$$g(x) = \frac{1}{2}\sum_i c_i(x_i + \overline{y_i}) + \frac{1}{2}\sum_{i<j} -c_{ij}^-(x_i x_j + \overline{y_i y_j}) + c_{ij}^+(x_i \overline{y_j} + \overline{y_i} x_j)$$

satisfies $g(x, \overline{x}) = f(x)$ and it gives an optimal submodular relaxation to f .

Here, $c_{ij}^- = -\min(c_{ij}, 0),\ \ c_{ij}^+ = \max(c_{ij}, 0).$

Partial Assignment from Roof Dual

Let $(x^*, y^*) \in S^n$ be a solution of the roof dual optimization problem

$$\min_{(x,y)\in\{0,1\}^{2n}} g(x,y).$$

Then we construct a partial assignment as follows:

- Let $S \subset \{1,\ldots,n\}$ be defined by

$$S = \{i \in \{1,\ldots,n\} \mid (x_i^*, y_i^*) \neq (0,0)\}$$

- Define $z_S \in \{0,1\}^S$ by $(z_S)_i = x_i^*,\ \ i \in S$.

Persistency

Theorem: Weak persistency holds at z_S, that is, there exists an extension $z \in \{0,1\}^n$ of z_S with

$$z \in \mathrm{argmin}_x\, f(x).$$

Summary

- Given an arbitrary quadratic pseudo-Boolean function in n variables, there exists a submodular function g in $2n$ which leads to an optimal lower bound on the original function.
- A minimizer of g can be computed using graph cuts as seen in the last lecture.
- From a minimizer of g, a partial assignment for the original optimization problem can be obtained. Weak persistency holds for this partial assignment.