Variational Methods for Computer Vision

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TU Munich, Fall 11/12

Some Literature

Literature on Variational Methods

P. Kornprobst, G. Aubert, “Mathematical Problems in Image Processing, Partial Differential Equations and the Calculus of Variations”, Springer 2006.

T. Chan, J. Shen, “Image Processing and Analysis: Variational, PDE, Wavelet, and Stochastic Methods”, SIAM 2005.

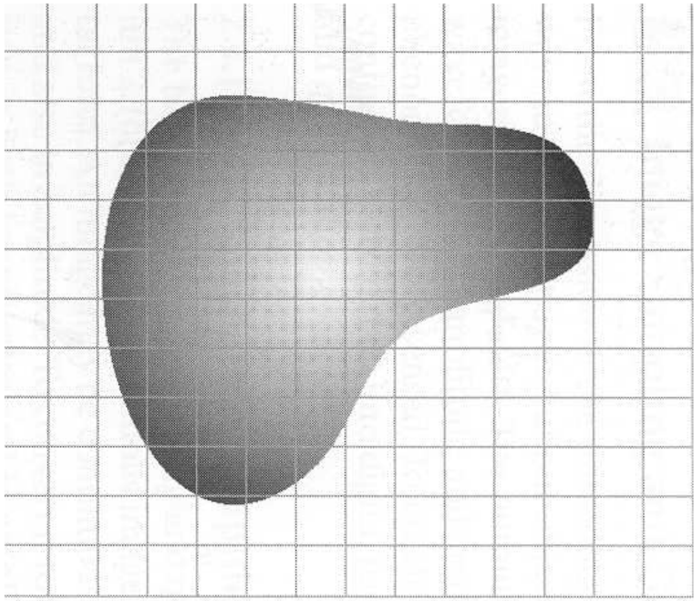
J.-M. Morel, S. Solimini, “Variational Methods in Image Segmentation”, Birkhäuser 1995.

K. Bredies, D. Lorenz, “Mathematische Bildverarbeitung: Einführung in Grundlagen und moderne Theorie”, Vieweg & Teubner 2011.

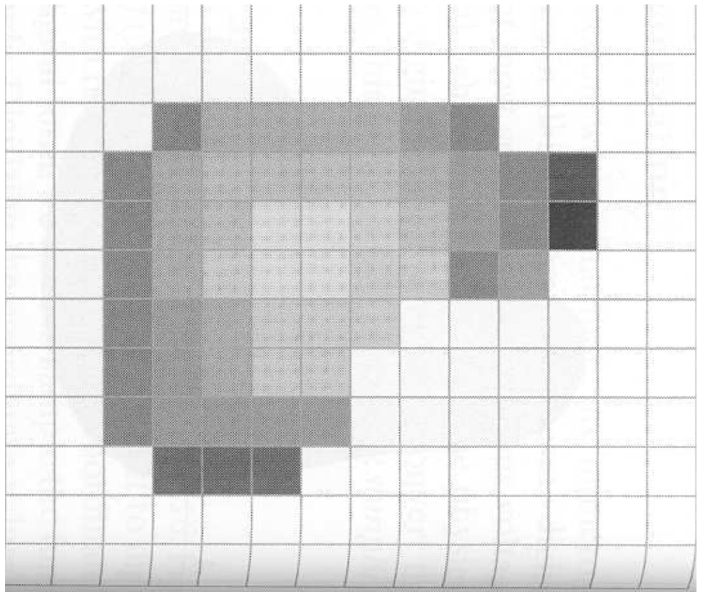
The Continuum Representation

Continuous versus Discrete

Digital images are discrete, both in space and in their values. Nevertheless, one can treat represent and analyze them in a continuous setting.



continuous



discrete
(sampling & quantization)

Continuous versus Discrete

- There are different levels of discretization:
  - Discretization in color or brightness space (=quantization)
  - Discretization in (physical) space
  - Discretization in time (for videos)
- Continuous representation: $f : (\Omega \subset \mathbb{R}^n) \to \mathbb{R}^d$
- $n = 2$: 2-dim. images,
  $n = 3$: volumetric images or 2-dim. videos,
  $n = 4$: volume + time,...
- $d = 1$: brightness images,
  $d = 3$: color images,
  $d > 1$: multispectral images
- Discretization:

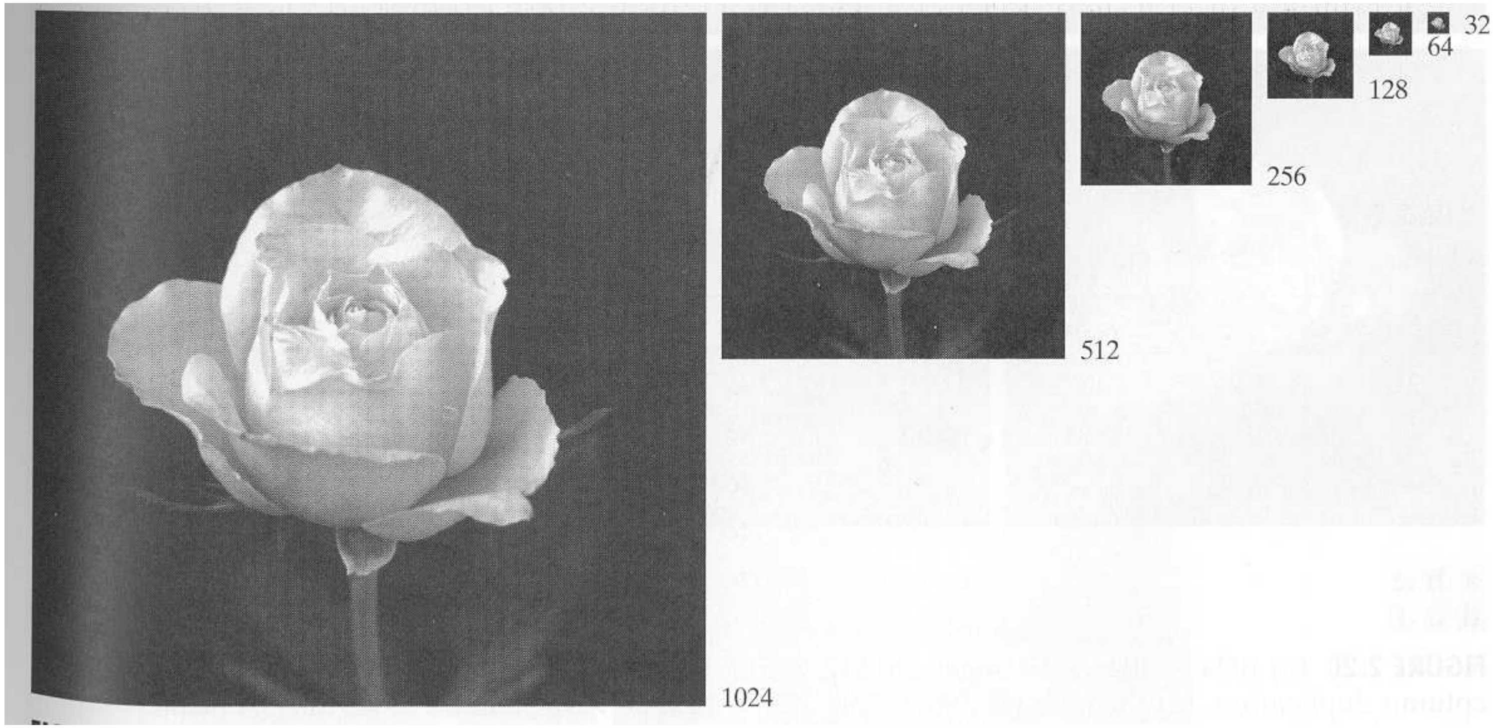
$$f(x,y) \longrightarrow \begin{bmatrix} f(1,1) & f(1,2) & \cdots & f(1,N) \\ f(2,1) & f(2,2) & \cdots & f(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ f(M,1) & f(M,2) & \cdots & f(M,N) \end{bmatrix}$$

Continuous versus Discrete

- Advantages of discrete representations:
  - Digital images *are* discrete, and their processing in a computer will ultimately require a discretization.
  - No numerical approximations in modeling the transition from discrete to continuous.
  - For various problems there exist efficient algorithms from discrete optimization.

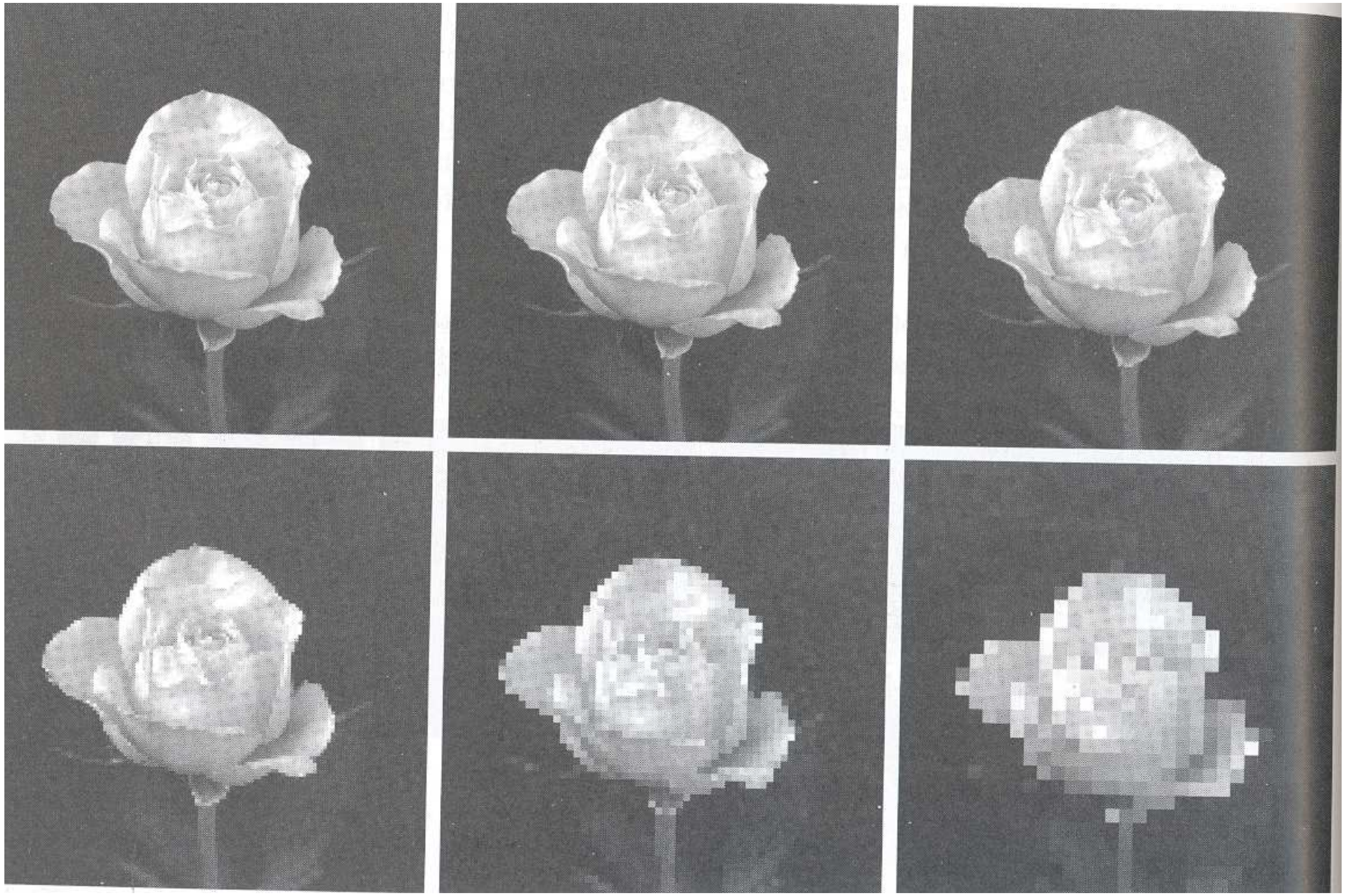
- Advantages of continuous representations:
  - The world observed through the camera is continuous.
  - There exists abundant mathematical theory for the treatment of continuous functions (functional analysis, differential geometry, partial differential equations, group theory,...).
  - Certain properties (rotational invariance) are easier to model because artefacts of discretization can be ignored.
  - Continuous models correspond to the limit of infinitely fine discretization.

Spatial Subsampling



Representation of an image with fewer and fewer pixels
(source: Gonzalez & Woods)

Spatial Subsampling

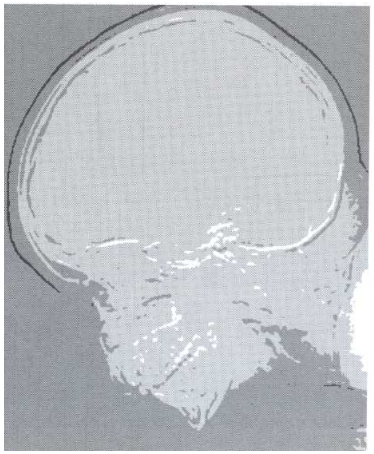


Subsampled from 1024^2 to 32^2 and enlarged
(Quelle: Gonzalez & Woods)

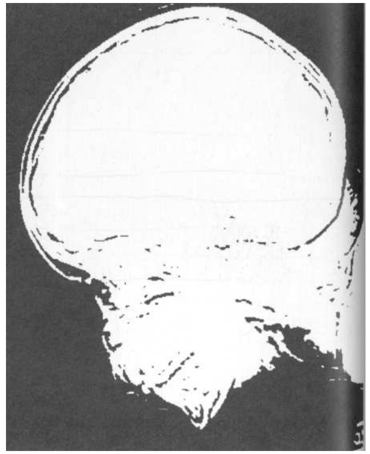
Brightness Quantization

256 levels

16 levels



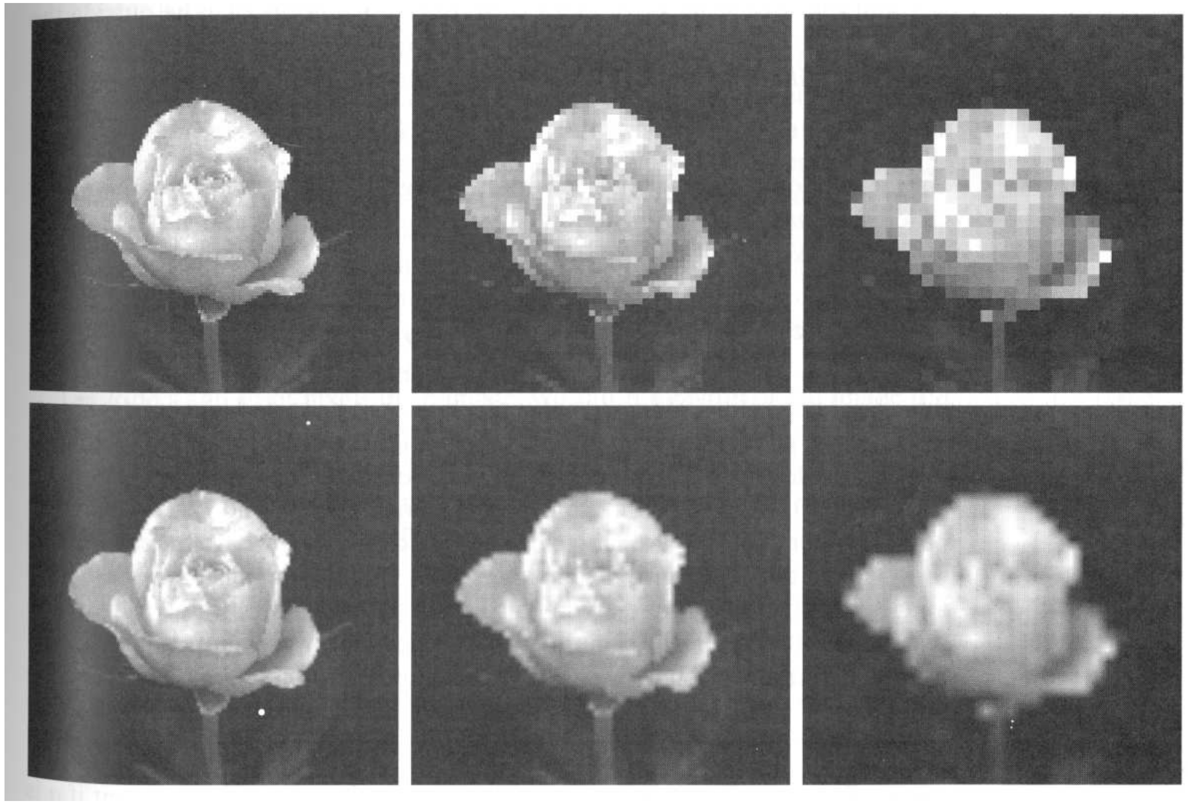
4 levels



2 levels

“typical” images: 256×256 pixels with 256 brightness values

Interpolation



lower row: Bilinear interpolation of the upper row

Bilinear interpolation: $\hat{f}(x,y) = ax + by + cxy + d$ with coefficients a, b, c, d determined by fitting to brightness values of 4 neighboring pixels.

Alternatives: nearest neighbor interpolation, bi-cubic interpolation,...

Spatial Domain Filtering

Spatial Domain Filtering

- Filtering of an image in the spatial domain can be represented by an operator T:

$$g(x,y) = (Tf)(x,y),$$

where f denotes the input image and g the processed image.

- Typically T acts on a certain spatial neighborhood.
- The simplest form of T is an operator which simply models a local brightness transformation:

$$s = T(r),$$

where r is the input brightness at a certain location and s the respective brightness in the transformed image.

The Brightness Transform

- Typically the goal of brightness transforms is to transfer the brightness values into a range that facilitates it for humans to see the relevant structures, i.e. the semantically important brighness transitions should be in a range where retinal receptors are particularly sensitive.

- In most cases one considers monotonically nondecreasing brightness transforms $T(r)$, i.e. transforms which preserve the ordering:

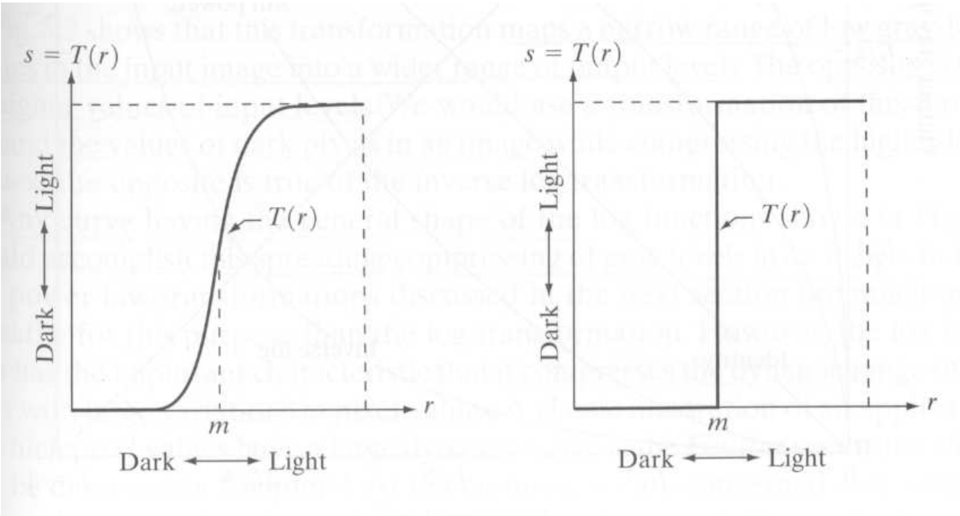
$$r_1 \leq r_2 \Rightarrow T(r_1) \leq T(r_2)$$

  In the case that $r_1 < r_2 \Rightarrow T(r_1) < T(r_2)$ these transforms are called strictly monotonous.

- Strictly monotonous brightness transforms are invertible, i.e. the orginal image data can be recovered from the filtered image.

Contrast enhancement

- Two important examples of brightness transforms are contrast stretching (Kontrastverstärkung) and thresholding (Schwellwertbildung):

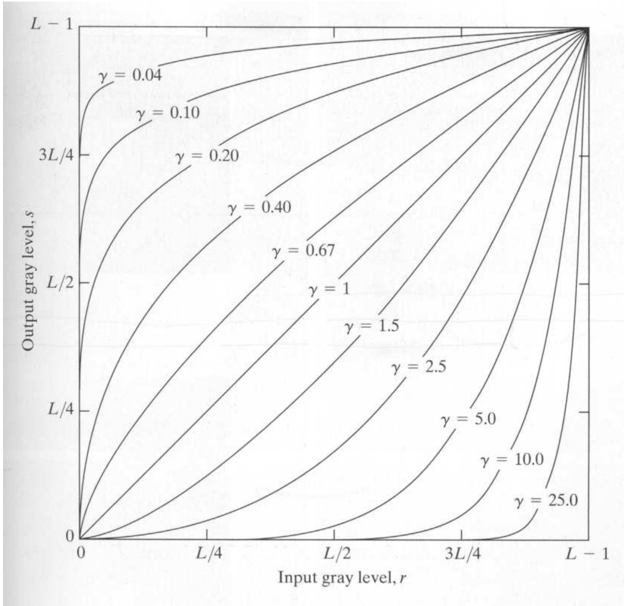


contrast stretching and thresholding
(Quelle: Gonzalez & Woods)

- Thresholding can be seen as a limiting case of contrast stretching. It provides a binary image as output which is often useful for further processing.

Log- and Powerlaw-Transform

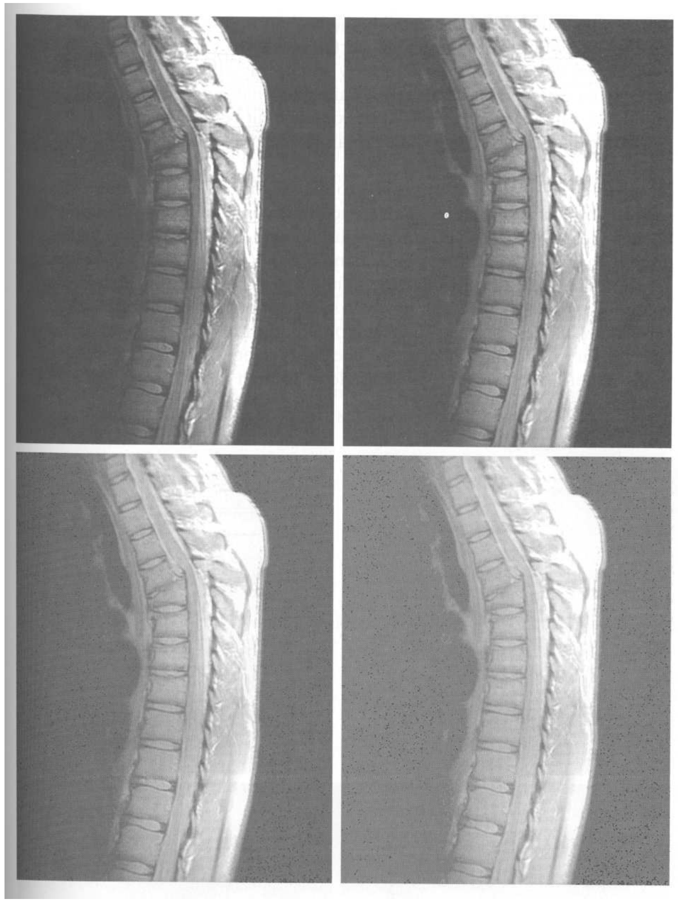
- Two further examples of brightness transforms are the logarithm transform: $s = c\log(1+r)$, and the powerlaw transform: $s = cr^\gamma$.



Powerlaw transform $s = cr^\gamma$

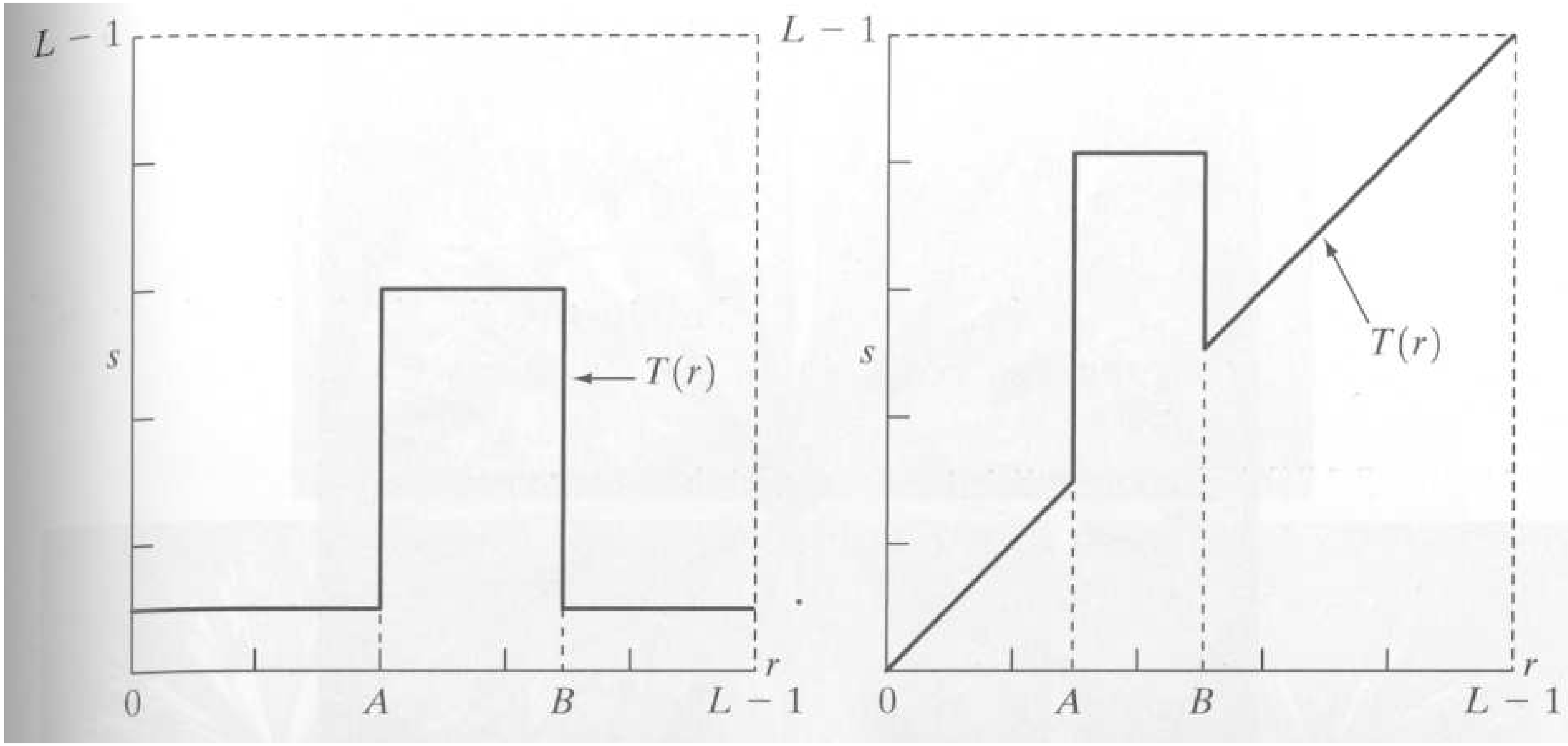
- Nonlinear brightening ($\gamma < 1$) or darkening ($\gamma > 1$).
- The correction of brightness changes (due to image acquisition and image display) with an inverse powerlow transform is called gamma correction.

Example: Contrast Enhancement



Input image (upper left) and powerlaw transforms with $\gamma = 0.6, 0.4$ and 0.3. Through the powerlaw transform certain image regions become more visible.

Gray Level Slicing



What effects do the above transforms have?

Linear Filters

- The term filtering is derived from frequency space methods where a spatial smoothing of the brightness values corresponds to a signal transform where high-frequency components are filtered out.
- An operator T is called linear if the following properties hold:

  1. $T(f+g) = T(f) + T(g) \quad \forall$ images f, g.
  2. $T(\alpha f) = \alpha T(f) \quad \forall$ images f, scalars α.

- For linear operators, the output brightness values are linear combinations of the input brightness values. Among the linear transformations is the convolution (Faltung):

$$g(x,y) = \int w(x',y')\, f(x-x',y-y')\, dx'\, dy'.$$

  In a spatially discrete setting, this corresponds to a weighted sum:

$$g(i,j) = \sum_{m,n} w(m,n)\, f(i-m, j-n).$$

Linear Filters

- In practice this summation extends over a certain neighborhood, often called window. The matrix of weights $w(m,n)$ is called a mask.

$$g(i,j) = \sum_{m,n} w(m,n)\, f(i-m, j-n)$$

- For example, the 3×3 mask:

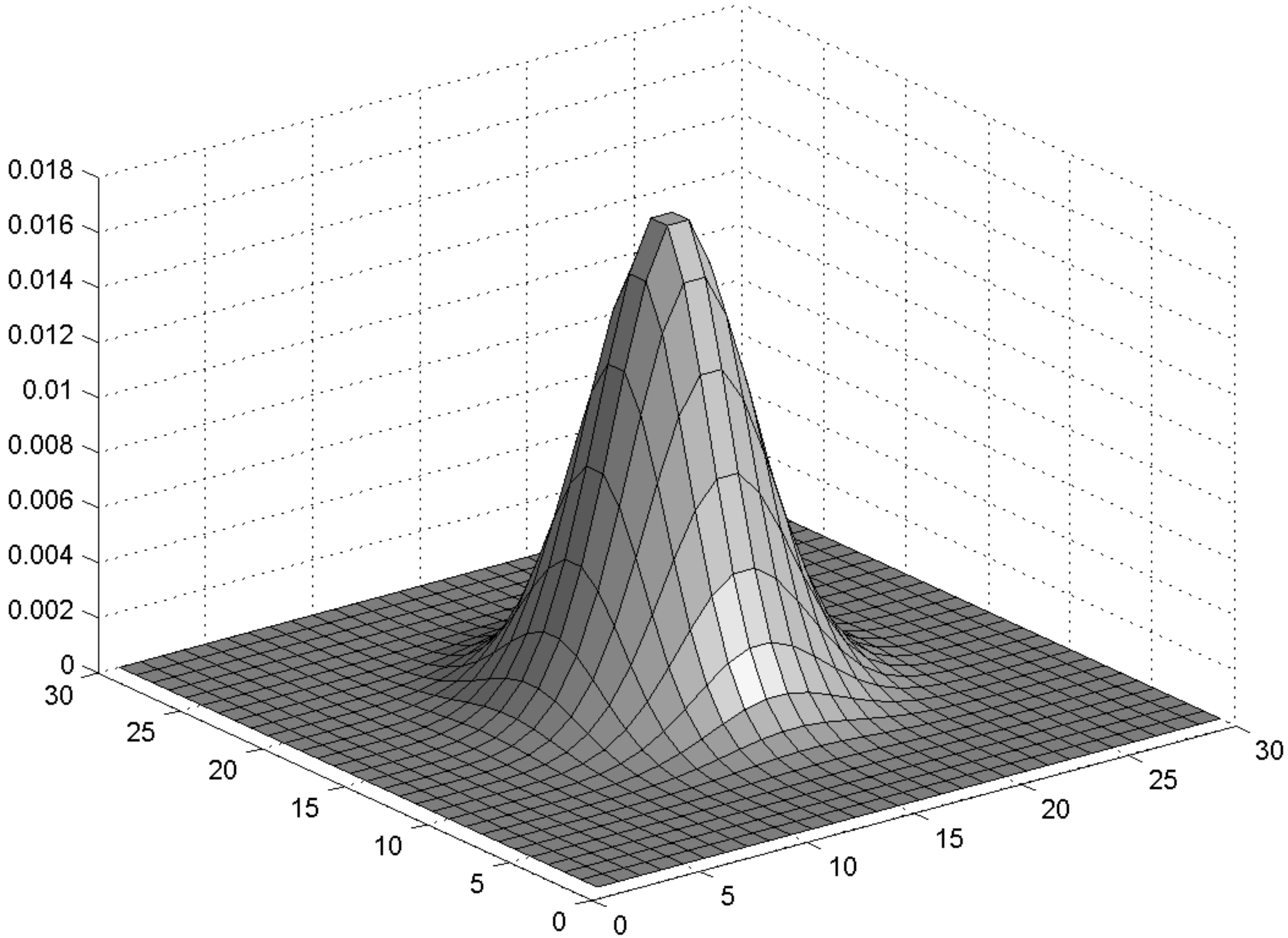
w(1,1)	w(0,1)	w(-1,1)
w(1,0)	w(0,0)	w(-1,0)
w(1,-1)	w(0,-1)	w(-1,-1)

- In the continuous representation the weight function $w(x',y')$ is called convolution kernel (Faltungskern):

$$g(x,y) = (w * f)(x,y) \equiv \int w(x',y')\, f(x-x', y-y')\, dx'\, dy'$$

Gaussian Convolution

$w(x,y)$



Smoothing Filters

- Smoothing, averaging or low-pass filters typically average the brightness values in a certain spatial neighborhood.
- The most common example of smoothing kernel is the Gaussian kernel. It induces a weighted average of brightness values on the scale determined by the standard deviation σ:

$$w(x,y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

- A multitude of alternative convolution kernels (or filter masks) is conceivable, for example box filters which are constant within the window:

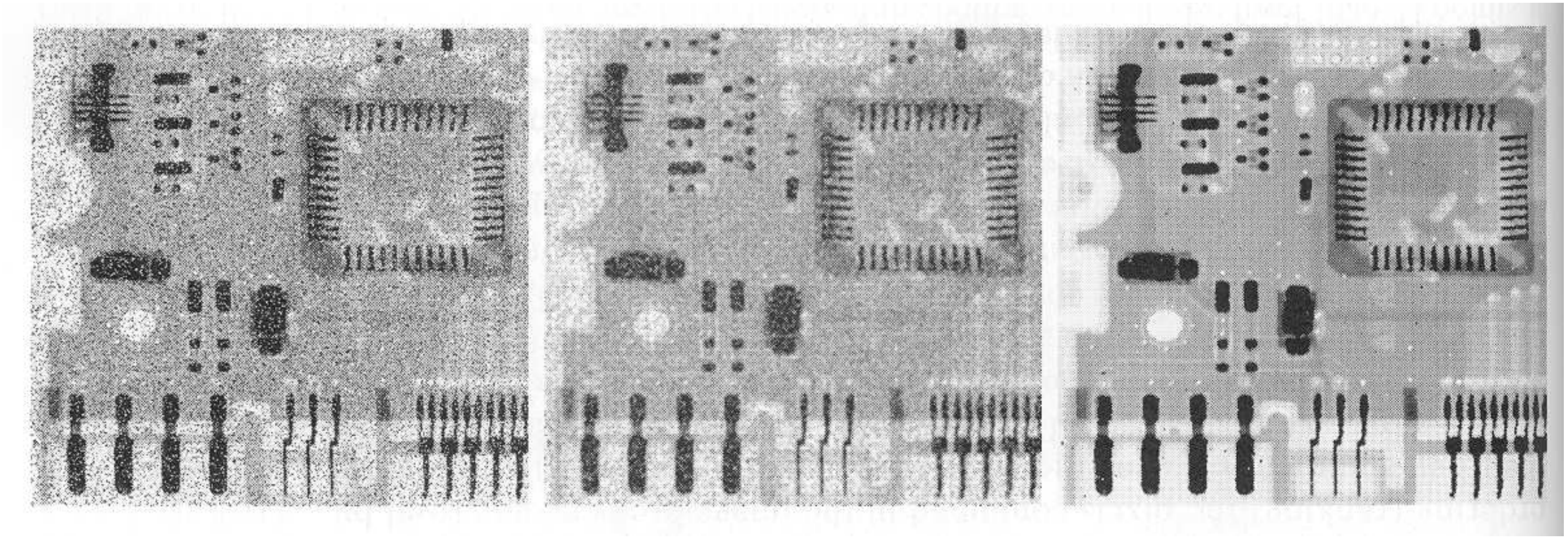
$$w(i,j) = \frac{1}{9} \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

- For pixels at the image boundary, the weight mask must be adapted appropriately.

The Median Filter

- A specific class of nonlinear filters are the order statistics filters. For these filters, the brightness of the filtered image at a given pixel depends on the order of brightness values in a certain neighborhood.
- The best known example of an order statistics filter is the median filter. For this filter, each pixel is assigned the median value of brightness values in its neighborhood.
- Example: The median of the brightness values $\{1, 2, 2, 3, 4, 5, 20\}$ ist 3, i.e. the central value after sorting.
- Median filters are particularly useful for reduction of impulse noise, also called salt-and-pepper noise, i.e. noise where some brightness values are randomly replaced by black or white values.
- Median filters typically induce less blurring than Gaussian or other linear smoothing filters.

Median versus Gauss



noisy input | Gauss filtered | median filtered

In contrast to the Gaussian filter (center), the median filter better removes noise without blurring structures. Nonlinear methods are often more general and more powerful than linear approaches.

Derivative Filters

- Derivative filters capture the spatial variations of brightness. In particular, they provide information about edges or corners in an image. In a simplified world of black objects on white ground, these brightness edges correspond to object boundaries.

- Mathematically the partial derivatives of the function $f(x,y)$ with respect to x is defined as:

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon,y) - f(x,y)}{\varepsilon}$$

- This continuous derivative of a function in x-direction can be approximated discretely by (symmetric) finite differences:

$$\partial_x f(x,y) \equiv f_x(x,y) \equiv \frac{\partial f(x,y)}{\partial x} \approx \frac{f(x+1,y) - f(x-1,y)}{2}$$

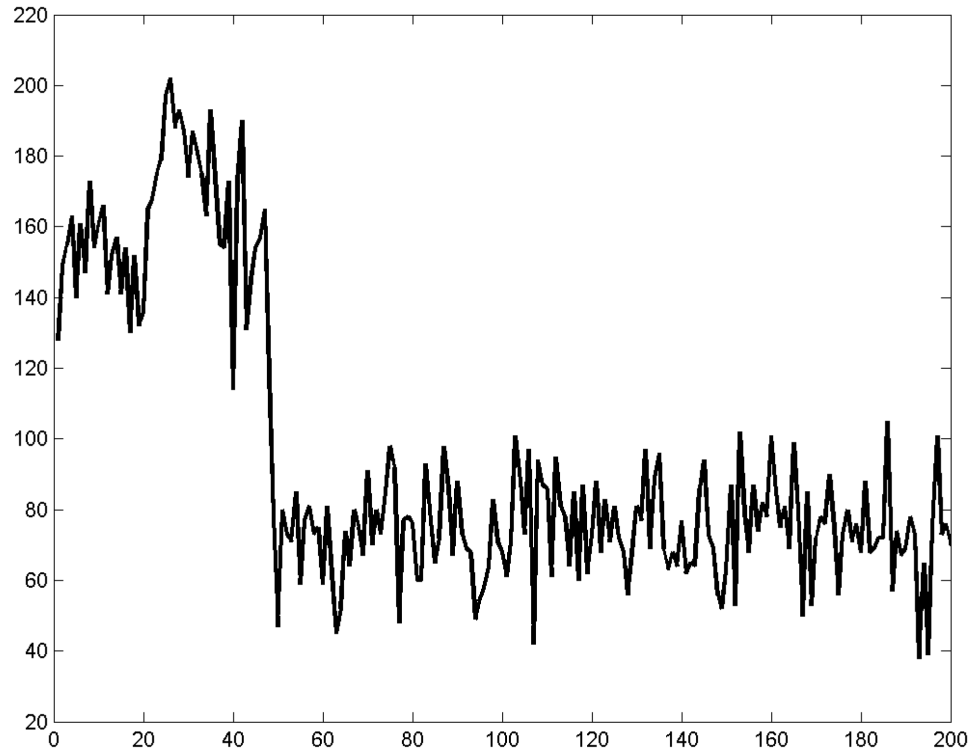
- Alternatives:

$$\partial_x f(x,y) \approx f(x+1,y) - f(x,y) \text{ (Forward difference)}$$

$$\partial_x f(x,y) \approx f(x,y) - f(x-1,y) \text{ (Backward difference).}$$

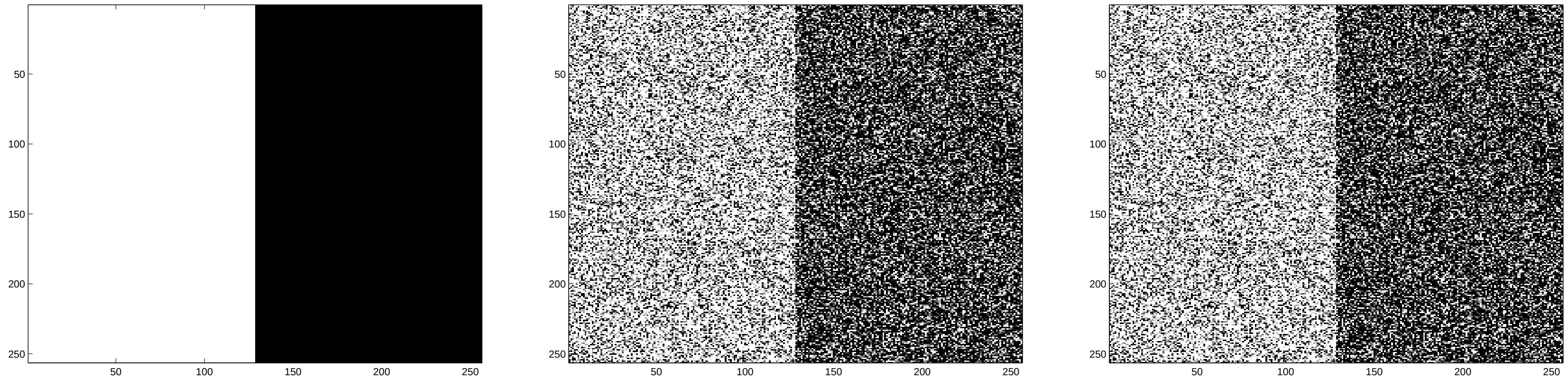
Example: 1D Brightness Profile

Input

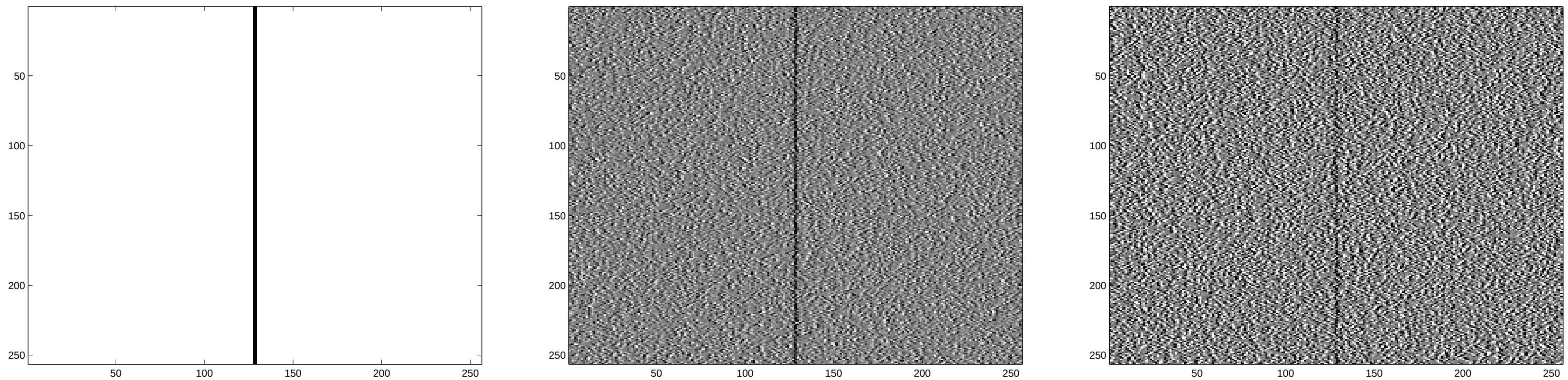


1D brightness profile

Example of the First Derivative

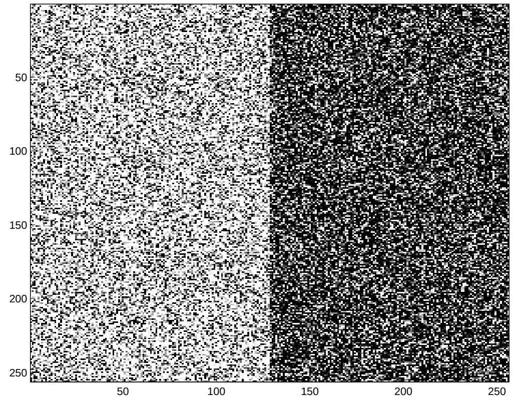


Input images f (with noise)

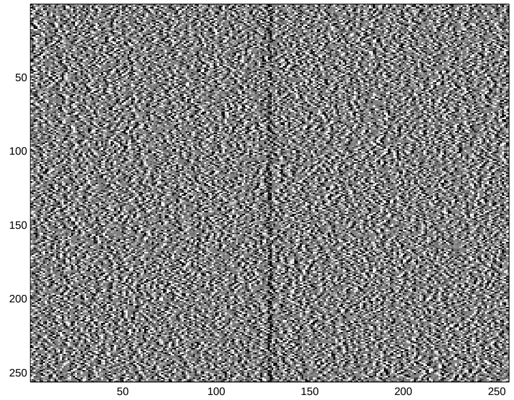


First derivative in x-direction f_x

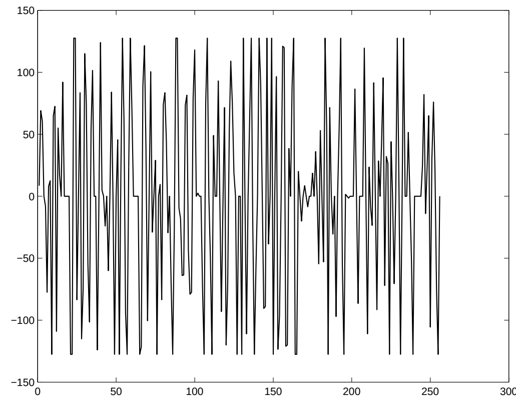
Noise Sensitivity of the Derivative



Input f



derivative f_x



f_x along horizontal line

Observation:

- Vertical edges can be determined as maxima of the norm of the x-derivative.
- Horizontal edges can be determined as maxima of the norm of the y-derivative.
- This approach only allows to selectively determine horizontal or vertical edges.
- It is very sensitive to noise.

The Image Gradient $\nabla f(x,y)$

- The gradient of a function $f(x,y)$ is the vector:

$$\nabla f(x,y) = \begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} \equiv \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

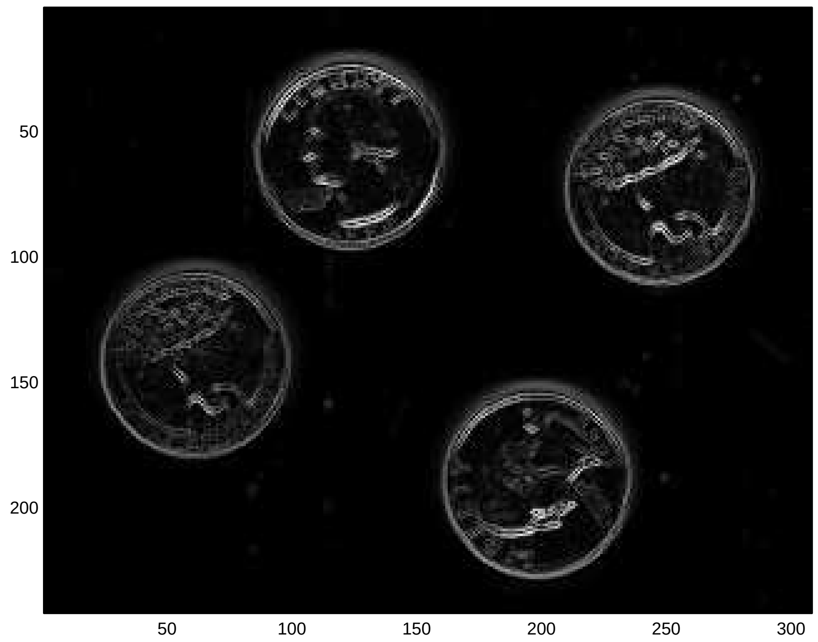
- The gradient norm (often also called “gradient”) is given by the length of the gradient vector:

$$|\nabla f(x,y)| = \left| \begin{pmatrix} f_x \\ f_y \end{pmatrix} \right| = \sqrt{(f_x)^2 + (f_y)^2}$$

- Der gradient norm is a nonlinear operator operator for detection of edges in arbitrary orientation.
- The gradient norm is rotationally covariant (sometimes called rotationally invariant). This means: The gradient norm of the rotated image is the same as the rotated gradient norm of the unrotated image. This implies that the performance of this operator does not depend on how the input image is rotated.

Example of the Image Gradient

Input image



Gradient norm

The Laplace Operator $\Delta f(x,y)$

- The divergence of a vector $v = (v_1, v_2)$ is defined as $\nabla v = \partial_x v_1 + \partial_y v_2$.
- The Laplace operator Δ is given by the concatenation of gradient and divergence:

$$\Delta f(x,y) = \nabla^2 f(x,y) = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f_{xx} + f_{yy}$$

- The Laplace operator is linear:

$$\Delta(\alpha_1 f(x) + \alpha_2 g(x)) = \alpha_1 \Delta f(x) + \alpha_2 \Delta g(x) \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \forall f, g$$

- Linearity has several practical advantages. Linearity implies that it does not matter whether one first sums images and then processes them or vice versa.

Example of the Laplace Operator

Input image

Laplace operator of the image

Discretization of Derivatives

There exist different discrete approximations of derivatives. In the following we shall denote width and height of a single pixel by h_x and h_y. Then the x-derivative of a brightness image f at pixel (i,j) can be approximated as:

1. Symmetric differences:

$$f_x(i,j) \approx \frac{f(i+1,j) - f(i-1,j)}{2h_x}$$

2. Forward differences:

$$f_x(i,j) \approx \frac{f(i+1,j) - f(i,j)}{h_x}$$

3. Backward differences:

$$f_x(i,j) \approx \frac{f(i,j) - f(i-1,j)}{h_x}$$

How do these masks differ? Which one is better?

The Taylor Series Expansion

- The key idea of the Taylor expansion is to approximate a function in the vicinity of an expansion point by a truncated power series:

$$f(x_0+\varepsilon) = f(x_0) + \varepsilon f'(x_0) + \frac{\varepsilon^2}{2} f''(x_0) + O\left[\varepsilon^3\right]$$

- This expansion is easily derived as follows. Assume that the function $f(x_0+x)$ can be written as a linear combination of powers of x:

$$f(x_0+x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n$$

By inserting $x=0$ into various derivatives of this expression, we get:

$$f(x_0) = a_0, \quad f'(x_0) = a_1, \quad f''(x_0) = 2a_2$$

and in general:

$$f^{(n)}(x_0) = (n!)a_n \quad \Rightarrow \quad a_n = \frac{f^{(n)}(x_0)}{n!}$$

Application to the Brightness Function

Let $f(i,j)$ denote the brightness at pixel (i,j), and h_x the width of a pixel. Then we have:

$$f(i+1,j) = f(i,j) + h_x f_x(i,j) + \frac{h_x^2}{2} f_{xx}(i,j) + O\left[h_x^3\right]$$

Similarly:

$$f(i-1,j) = f(i,j) - h_x f_x(i,j) + \frac{h_x^2}{2} f_{xx}(i,j) + O\left[h_x^3\right]$$

Subtracting both equations leads to:

$$f_x(i,j) = \frac{f(i+1,j) - f(i-1,j)}{2h_x} + O\left[h_x^2\right]$$

Simply subtracting $f(i,j)$ from the first equation leads to:

$$f_x(i,j) = \frac{f(i+1,j) - f(i,j)}{h_x} + O\left[h_x\right]$$

Discretization Error

- The difference between an analytical expression and its discrete representation is called discretization error.
- When discretizing a differential equation, the order of the discretization error is called order of consistency.
- The symmetric difference discretization is of order 2 because the discretization error is of order h_x^2. In contrast, forward or backward differences lead to a consistency order 1.
- In the numerical discretization of differential equations higher orders are typically better because they allow to faster approximate the continuum with finer discretizations, i.e. $h_x \to 0$.
- Using Taylor expansions of higher order one can further improve the consistency order, however at the sake of larger mask size.

Discretization of the 2nd Derivative

As above, let $f(i,j)$ denote the brightness at pixel (i,j), and h_x the width of each pixel. Then we have:

$$f(i+1,j) = f(i,j) + h_x f_x(i,j) + \frac{h_x^2}{2} f_{xx}(i,j) + \frac{h_x^3}{3!} f_{xxx}(i,j) + O\left[h_x^4\right]$$

Similarly:

$$f(i-1,j) = f(i,j) - h_x f_x(i,j) + \frac{h_x^2}{2} f_{xx}(i,j) - \frac{h_x^3}{3!} f_{xxx}(i,j) + O\left[h_x^4\right]$$

Summing both equations leads to:

$$f_{xx}(i,j) = \frac{f(i+1,j) + f(i-1,j) - 2f(i,j)}{h_x^2} + O\left[h_x^2\right]$$

This discretization of the second derivative is of consistency order 2.

Discretization of the Laplacian

0	1	0
1	−4	1
0	1	0

1	1	1
1	−8	1
1	1	1

Two masks showing discretizations of $\Delta f = f_{xx} + f_{yy}$.