TU München
Fakultät für Informatik
Dr. Claudia Nieuwenhuis
Eno Töppe

Statistical Methods and Learning in Computer Vision
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Homework Assignment 4 — November 21st, 2012

Exercise 1. (Unbiased Estimators for Sample Mean and Variance)

We want to find unbiased estimators for the mean and variance of a given sample set.

Unbiasedness

Let $\hat{g} : \mathbb{H} \to \Gamma$ be a function, which maps from the sample space $\mathbb{H}$ to the parameter space Γ containing the possible mean and variance values. Let P_γ be the assumed true distribution of the given samples. Then $\hat{g}$ is called an *unbiased* estimator for the parameter γ if the expection value of the estimator under the true distribution P_γ equals the parameter to be estimated

$$E_{P_\gamma}(\hat{g}(x)) = \gamma$$

For example, the mean μ of the Gaussian distribution can be estimated by the sample mean $\hat{g}(X) = \frac{1}{N}\sum_{i=1}^{N} X_i$. This estimator is unbiased, since $E(\hat{g}(X)) = \mu$.

Let

$$\xi_\gamma := E_{P_\gamma}(X), \sigma_\gamma^2 := V_{P_\gamma}(X)$$

denote the true expectation value and variance of the sample set.

a) Show that

$$\begin{aligned} \bar{X} &:= \frac{1}{n}\sum_{i=1}^{n} X_i \\ S^2 &:= \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \end{aligned}$$

are unbiased estimators for the sample mean and variance.

b) Show that the variance of the $\bar{X}$ estimator is given by $V_\gamma(\bar{X}) = \frac{1}{n}\sigma_\gamma^2$

c) Show that the maximum likelihood estimators of the mean and variance of the one-dimensional Gaussian distribution with density

$$f(z) = \frac{1}{\sqrt{2\pi}\sigma} \exp - \frac{(z-a)^2}{2\sigma^2}$$

are given by

$$
\begin{aligned}
a &= \frac{1}{n}\sum_{i=1}^{n} X_i \\
\sigma^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i - a)^2
\end{aligned}
$$

This demonstrates that for Gaussian distributions the maximum likelihood estimate is unbiased for the mean, but biased for the variance.

Exercise 2. (Principal Component Analysis)

We want to derive an alternative formulation of Principal Component Analysis (PCA) in terms of projection error minimization.
We start from a complete orthonormal set of D-dimensional basis vectors $\{u_i\}$ satisfying $u_i^T u_j = \delta_{ij}$. Since this basis is complete each data point can be represented exactly by a linear combination of the basis vectors

$$x_n = \sum_{i=1}^{D} \alpha_{ni} u_i = \sum_{i=1}^{D} (x_n^T u_i) u_i$$

For $M < D$ dimensions the linear subspace can be represented by the first M basis vectors. Each data point can now be approximated by

$$\tilde{x}_n = \sum_{i=1}^{M} z_{ni} u_i + \sum_{i=M+1}^{D} b_i u_i.$$

Here the z_{ni} depend on the particular data point, the b_i are constant for all data points. The second summand can be interpreted as a translation orthogonal to the M-Dimensional subspace. Our goal is to find $\{u_i\}, \{z_{ni}\}$ and $\{b_i\}$, which minimize the projection error of the data points onto the subspace. That means we want to minimize the distortion measure

$$J = \frac{1}{N}\sum_{n=1}^{N} \|x_n - \tilde{x}_n\|^2.$$

To do this, prove the following steps

a) Take the derivative of J with respect to z_{ni} and b_i.

b) Compute the projection error $x_n - \tilde{x}_n$ by using the previous solution for z_{ni} and b_i.

c) Introduce the projection error into the distortion measure and express J by means of the sample covariance matrix

$$S = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(x_i - \bar{x})^T$$

d) Show for a 2-dimensional subspace (D=2, M=1) that the solution to the minimization of J yields u_2 as the eigenvector to the smallest eigenvalue λ_2 of S and $J = \lambda_2$. To this end, we minimize the following function, which uses a Lagrange multiplier to ensure basis vectors of length 1

$$\tilde{J} = u_2^T S u_2 + \lambda_2 (1 - u_2^T u_2)$$

This result can be generalized to D dimensions: $J = \sum_{i=M+1}^{D} \lambda_i$.

This shows that we minimize the projection error by removing the eigenvectors with the smallest eigenvalues from the orthogonal basis.

The next exercise class will take place on **December 4th, 2012**.

For downloads of slides and of homework assignments and for further information on the course see

`http://vision.in.tum.de`