Image Evolutions

Image evolutions

Consider images which *evolve over time*

$$u : \Omega \times [0, T] \to \mathbb{R}^n$$

The image has now three parameters: $u(x, y, t)$.

Discretized view

Generate a *sequence* of images $u^k : \Omega \to \mathbb{R}^n$ starting with some u^0:

$$u^0, u^1, u^2, u^3, \ldots$$

by a specific algorithm. Only the result u^{k_0} for some $k_0 \geq 1$ is of interest.

Diffusion

We will first consider grayscale images $u : \Omega \times [0, T] \to \mathbb{R}$, and later generalize to multi-channel images.

Diffusion
Continuous-time update equation

$$\partial_t u = \operatorname{div}(D \nabla u)$$

Starting with some image $u(t = 0) = u^0$, this tells how the image must be changed over time. ∇ and div are only w.r.t. spatial variables x, y.

Diffusion tensor
$D : \Omega \times [0, T] \to \mathbb{R}^{2 \times 2}$ is called the *diffusion tensor*. It gives a *symmetric, positive definite* 2×2 *matrix* $D(x, y, t)$ for all (x, y, t). It may be different for every (x, y, t), and may depend on u.

Intuitively
Diffusion tries to *locally* cancel out any existing color differences, the image u gradually becomes *more and more smooth* over time.

Diffusion: Computation of the Update

Diffusion

$$(\partial_t u)(x,y,t) = \big(\operatorname{div}(D\nabla u)\big)(x,y,t)$$

1. Start with image $u : \Omega \times [0,T] \to \mathbb{R}$, values $u(x,y,t) \in \mathbb{R}$
2. Compute the gradient

$$g(x,y,t) := (\nabla u)(x,y,t) = \begin{pmatrix} (\partial_x u)(x,y,t) \\ (\partial_y u)(x,y,t) \end{pmatrix} \in \mathbb{R}^2$$

3. Multiply the diffusion tensor $D(x,y,t) \in \mathbb{R}^{2\times 2}$ with the gradient $g(x,y,t) \in \mathbb{R}^2$:

$$v(x,y,t) := D(x,y,t)g(x,y,t) \in \mathbb{R}^2$$

4. Take divergence of v:

$$d(x,y,t) := (\operatorname{div} v)(x,y,t) = (\partial_x v_1)(x,y,t) + (\partial_y v_2)(x,y,t) \in \mathbb{R}$$

Diffusion: Types

Diffusion

$$\partial_t u = \operatorname{div}(D \nabla u)$$

Linear/Nonlinear

- Linear: D *does not* depend on u
- Nonlinear: D depends on u

Additivity property of *linear* diffusion:
Given the solutions u and v for starting images u^0 and v^0, respectively, the solution for the starting image $u^0 + v^0$ is given by $u + v$.

Diffusion: Types

Diffusion

$$\partial_t u = \operatorname{div}(D \nabla u)$$

Isotropic/Nonisotropic

- Isotropic: Diffusivity matrix D is a scaled identity matrix:

  $$D(x, y, t) = g(x, y, t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

  $g(x, y, t) \in \mathbb{R}$ is called **diffusivity**. The diffusion equation becomes

  $$\partial_t u = \operatorname{div}(g \nabla u)$$

- Anisotropic: Any diffusion which is not isotropic.

Isotropic diffusion spreads out the values u *equally in every direction.*

Anisotropic diffusion can selectively suppress information flow in certain directions, e.g. only smooth out u *along* potential edges, and *not across*.

Diffusion: Types

Each diffusion is either linear or nonlinear, and either isotropic or nonisotropic:

	isotropic	nonisotropic
linear	linear isotropic	linear nonisotropic
nonlinear	nonlinear isotropic	nonlinear nonisotropic

Example: Laplace Diffusion

Diffusion tensor is constant:

$$D(x, y, t) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Diffusion equation becomes

$$\partial_t u = \operatorname{div}(D \nabla u) = \Delta\, u$$

This is a *linear* and *isotropic* diffusion.

Effect: Blurry version of the input image
For $\Omega = \mathbb{R}^2$ one can show the explicit formula

$$u(x, y, t) = (G_{\sqrt{2t}} * u^0)(x, y).$$

This formula is **not** valid for rectangular domains Ω, only for $\Omega = \mathbb{R}^2$, but the Laplace diffusion results are still similar to Gaussian convolution.

Multi-channel images
Process channel-wise.

Example: Laplace Diffusion

Input at $t = 0$

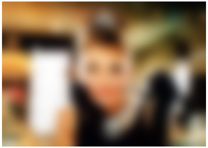
$t = 2$

$t = 4$

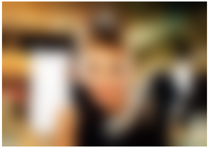
$t = 10$

$t = 20$

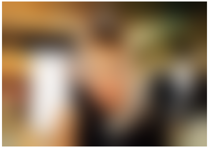
$t = 40$



$t = 100$



$t = 200$



$t = 400$

Example: Huber Diffusion

Diffusion tensor depends on the image u (or, more precicely, on ∇u):

$$D(x,y,t) = g(x,y,t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{with} \quad g(x,y,t) := \widehat{g}\Big(|\nabla u(x,y,t)|\Big) \quad \text{and} \quad \widehat{g}(s) := \frac{1}{\max(\varepsilon, s)}.$$

Diffusion equation becomes

$$\partial_t u = \operatorname{div}(D \nabla u) = \operatorname{div}\Big(\widehat{g}(|\nabla u|)\nabla u\Big)$$

This is a *nonlinear* and *isotropic* diffusion.

Effect: Smoothing with better edge preservation:
Edges of u are points (x,y) with large gradient value $|\nabla u(x,y)|$.
The diffusivity g is small in these points, so there will be less smoothing.

Multi-channel images
Channel-wise, but with *one common* diffusivity $\widehat{g}(|\nabla u|)$ for *all* channels.

Example: Huber Diffusion with $\varepsilon = 0.01$

Input at $t = 0$

$t = 0.04$

0.1

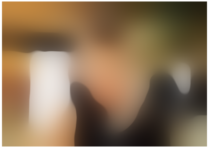
$t = 0.2$

$t = 0.4$

$t = 1$

$t = 2$

$t = 4$



$t = 10$

Discretization: General Isotropic Diffusion

Temporal derivative

Forward differences for ∂_t with a time step $\tau > 0$:

$$(\partial_t^+ u)(x, y, t) = \frac{u(x, y, t + \tau) - u(x, y, t)}{\tau}$$

Spatial derivatives

Forward differences for ∇, backward differences for div:

$$\mathrm{div}^- \left(g\, \nabla^+ u \right) = \partial_x^- \left(g\, \partial_x^+ u \right) + \partial_y^- \left(g\, \partial_y^+ u \right)$$

Diffusivity

Forward differences:

$$g = \widehat{g}\left(|\nabla^+ u| \right)$$

The current image $u(t)$ is used to compute g.

Discretization: General Isotropic Diffusion

Final scheme for general isotropic diffusion

$$u(x,y,t+\tau) = u(x,y,t) + \tau \,\mathrm{div}^{-}\Big(g\,\nabla^{+}u\Big) \qquad \text{with} \quad g = \widehat{g}\big(|\nabla^{+}u|\big)$$

Computation in several steps

1. Compute the gradient $G := \nabla^{+}u$
2. Compute the diffusivity $g = \widehat{g}(|G|)$
3. Compute the product $P := g \cdot G$
4. Compute the divergence $\mathrm{div}^{-}(P)$
5. Multiply by τ and add to u

Time step restriction
Only small τ possible. For monotonically decreasing $\widehat{g}$: $\tau < 0.25/\widehat{g}(0)$.

Discretization: Laplace Diffusion

Two ways to discretize the special case of Laplace diffusion, i.e. $g = 1$.

Final scheme for Laplace diffusion: Multi-step
One way is to use the above general multi-step procedure.

Final scheme for Laplace diffusion: Direct
Another way is to compute the update $\operatorname{div}^-(\nabla^+ u) = \Delta\, u$ directly in a single step, using the discretization from the previous lecture:

$$u(x, y, t+\tau) = u(x, y, t) + \tau\, (\Delta\, u)(x, y, t)$$

with

$$\begin{aligned}(\Delta\, u)(x, y, t) = \quad & \mathbf{1}_{x+1<W} \ \cdot\ u(x+1, y, t) \ + \ \mathbf{1}_{x>0} \ \cdot\ u(x-1, y, t) \\ & + \mathbf{1}_{y+1<H} \ \cdot\ u(x, y+1, t) \ + \ \mathbf{1}_{y>0} \ \cdot\ u(x, y-1, t) \\ & - \Big((\mathbf{1}_{x+1<W}) + (\mathbf{1}_{y+1<H}) + (\mathbf{1}_{x>0}) + (\mathbf{1}_{y>0}) \Big) \cdot u(x, y, t).\end{aligned}$$

Time step restriction
Only small τ possible: $\tau < 0.25$.