The EP Algorithm

- Given: a joint distribution over data and variables

$$p(\mathcal{D}, \boldsymbol{\theta}) = \prod_{i=1}^{M} f_i(\boldsymbol{\theta})$$

- Goal: approximate the posterior $p(\boldsymbol{\theta} \mid \mathcal{D})$ with q
- Initialize all approximating factors $\tilde{f}_i(\boldsymbol{\theta})$
- Initialize the posterior approximation $q(\boldsymbol{\theta}) \propto \prod_i \tilde{f}_i(\boldsymbol{\theta})$
- Do until convergence:
  - choose a factor $\tilde{f}_j(\boldsymbol{\theta})$
  - remove the factor from q by division: $q^{\setminus j}(\boldsymbol{\theta}) = \dfrac{q(\boldsymbol{\theta})}{\tilde{f}_j(\boldsymbol{\theta})}$

The EP Algorithm

- find q^{new} that minimizes

$$\mathrm{KL}\left(\frac{f_j(\theta)q^{\backslash j}(\boldsymbol{\theta})}{Z_j}\middle|q^{\text{new}}(\boldsymbol{\theta})\right)$$

  using moment matching, including the zero-th moment:

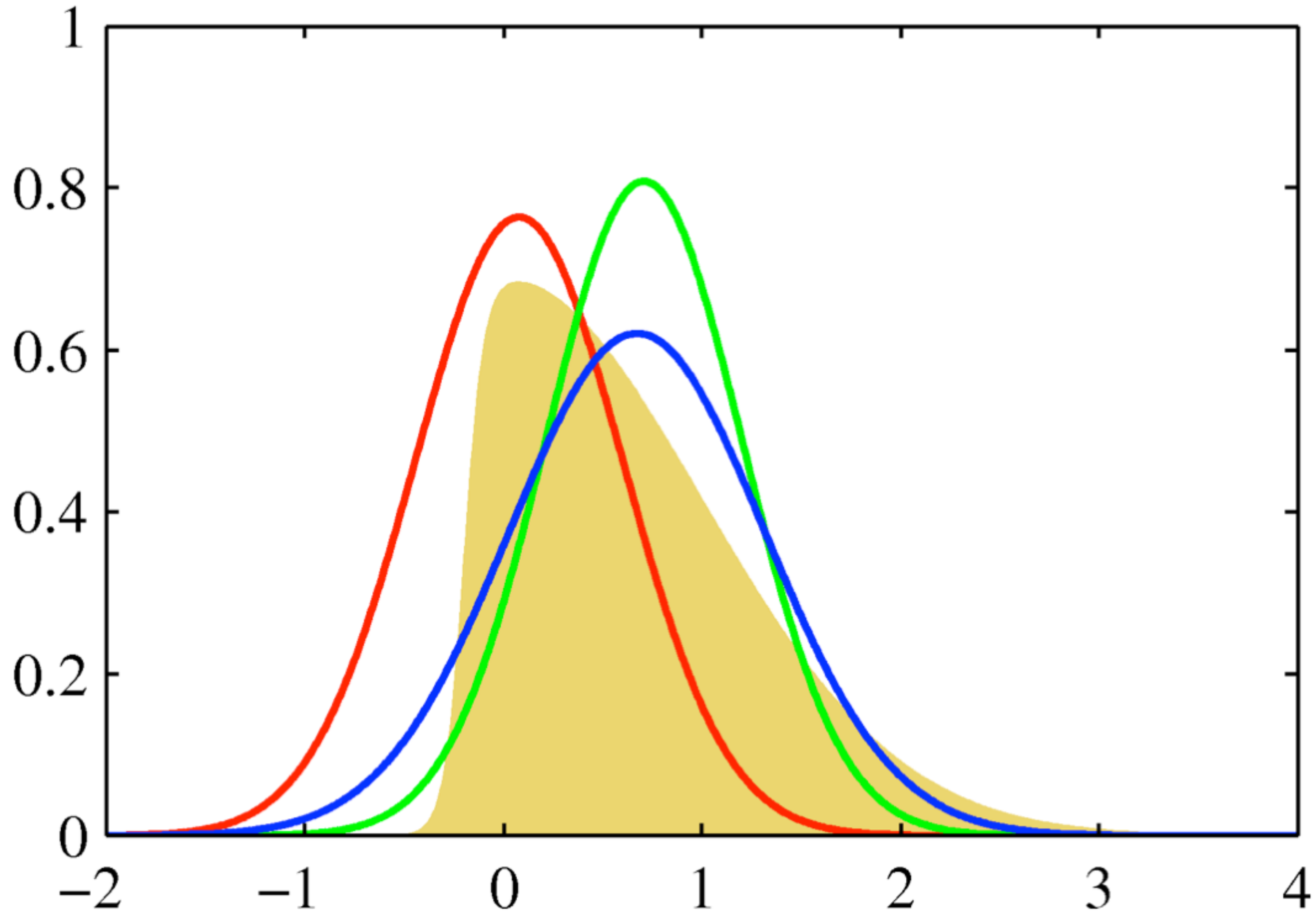
$$Z_j = \int q^{\backslash j}(\boldsymbol{\theta})f_j(\boldsymbol{\theta})d\boldsymbol{\theta}$$

- evaluate the new factor

$$\tilde{f}_j(\boldsymbol{\theta}) = Z_j \frac{q^{\text{new}}(\boldsymbol{\theta})}{q^{\backslash j}(\boldsymbol{\theta})}$$

- After convergence, we have $p(\mathcal{D}) \approx \int \prod_i \tilde{f}_j(\boldsymbol{\theta})d\boldsymbol{\theta}$

Example



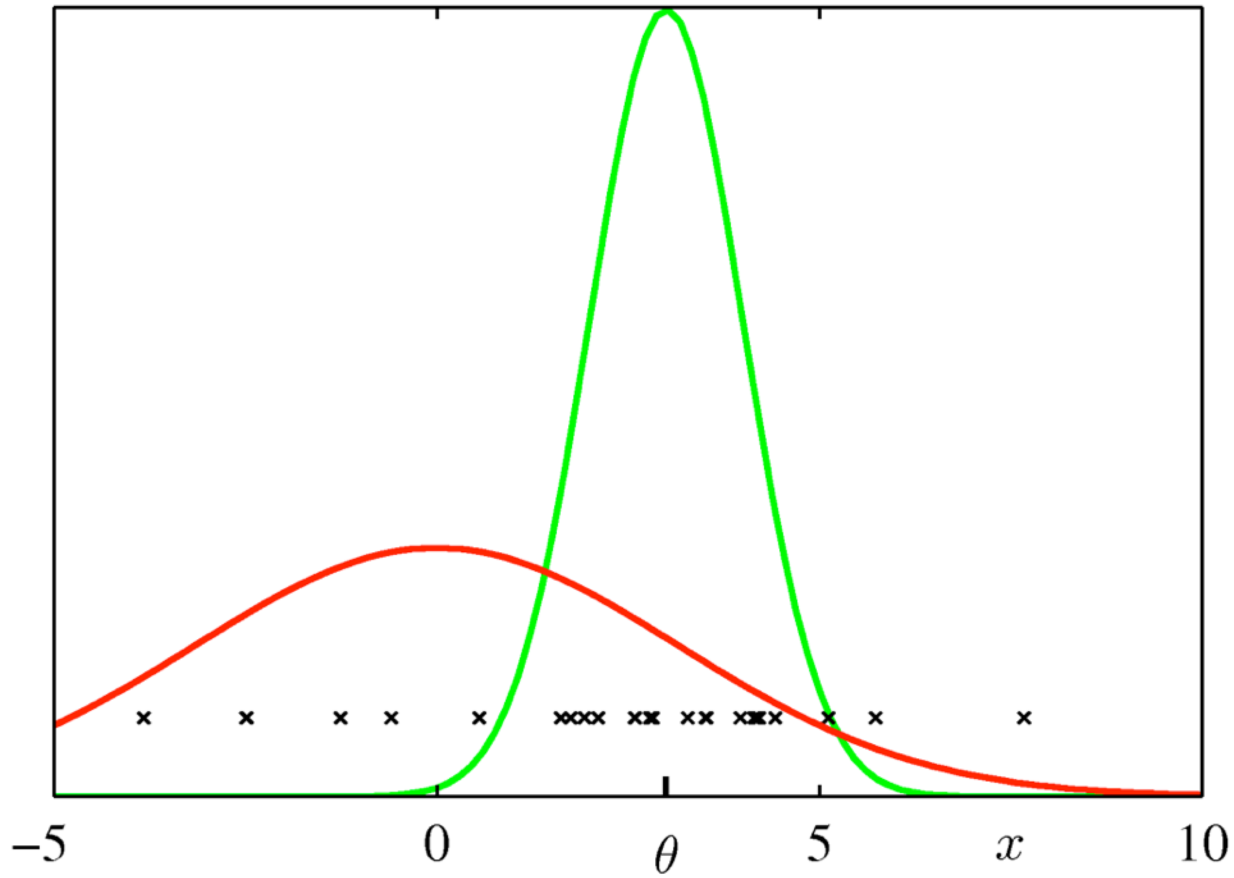
yellow: original distribution

red: Laplace approximation

green: global variation

blue: expectation-propagation

The Clutter Problem



- Aim: fit a multivariate Gaussian into data in the presence of background clutter (also Gaussian)

$$p(\mathbf{x} \mid \boldsymbol{\theta}) = (1-w)\mathcal{N}(\mathbf{x} \mid \boldsymbol{\theta}, I) + w\mathcal{N}(\mathbf{x} \mid \mathbf{0}, aI)$$

- The prior is Gaussian: $p(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta} \mid \mathbf{0}, bI)$

The Clutter Problem

The joint distribution for $\mathcal{D} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is

$$p(\mathcal{D}, \boldsymbol{\theta}) = p(\boldsymbol{\theta}) \prod_{n=1}^{N} p(\mathbf{x}_n \mid \boldsymbol{\theta})$$

this is a mixture of 2^N Gaussians! This is intractable for large N. Instead, we approximate it using a spherical Gaussian:

$$q(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta} \mid \mathbf{m}, vI) = \tilde{f}_0(\boldsymbol{\theta}) \prod_{n=1}^{N} \tilde{f}_n(\boldsymbol{\theta})$$

the factors are (unnormalized) Gaussians:

$$\tilde{f}_0(\boldsymbol{\theta}) = p(\boldsymbol{\theta}) \qquad \tilde{f}_n(\boldsymbol{\theta}) = s_n \mathcal{N}(\boldsymbol{\theta} \mid \mathbf{m}_n, v_n I)$$

EP for the Clutter Problem

- First, we initialize $\tilde{f}_n(\boldsymbol{\theta}) = 1$, i.e. $q(\boldsymbol{\theta}) = p(\boldsymbol{\theta})$
- Iterate:
  - Remove the current estimate of $\tilde{f}_n(\boldsymbol{\theta})$ from q by division of Gaussians:

$$q_{-n}(\boldsymbol{\theta}) = \frac{q(\boldsymbol{\theta})}{\tilde{f}_n(\boldsymbol{\theta})}$$

EP for the Clutter Problem

- First, we initialize $\tilde{f}_n(\boldsymbol{\theta}) = 1$, i.e. $q(\boldsymbol{\theta}) = p(\boldsymbol{\theta})$
- Iterate:
  - Remove the current estimate of $\tilde{f}_n(\boldsymbol{\theta})$ from q by division of Gaussians:

$$q_{-n}(\boldsymbol{\theta}) = \frac{q(\boldsymbol{\theta})}{\tilde{f}_n(\boldsymbol{\theta})} \qquad q_{-n}(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta} \mid \mathbf{m}_{-n}, v_{-n} I)$$

  - Compute the normalization constant:

$$Z_n = \int q_{-n}(\boldsymbol{\theta}) f_n(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

$$= (1-w)\mathcal{N}(\mathbf{x}_n \mid \mathbf{m}_{-n}, (v_{-n}+1)I) + w\mathcal{N}(\mathbf{x}_n \mid \mathbf{0}, aI)$$

EP for the Clutter Problem

- First, we initialize $\tilde{f}_n(\boldsymbol{\theta}) = 1$, i.e. $q(\boldsymbol{\theta}) = p(\boldsymbol{\theta})$
- Iterate:
  - Remove the current estimate of $\tilde{f}_n(\boldsymbol{\theta})$ from q by division of Gaussians:

$$q_{-n}(\boldsymbol{\theta}) = \frac{q(\boldsymbol{\theta})}{\tilde{f}_n(\boldsymbol{\theta})} \qquad q_{-n}(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta} \mid \mathbf{m}_{-n}, v_{-n} I)$$

  - Compute the normalization constant:

$$Z_n = \int q_{-n}(\boldsymbol{\theta}) f_n(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

  - Compute mean and variance of q^{new} using moment matching with $q_{-n}(\boldsymbol{\theta}) f_n(\boldsymbol{\theta})$

EP for the Clutter Problem

- First, we initialize $\tilde{f}_n(\boldsymbol{\theta}) = 1$, i.e. $q(\boldsymbol{\theta}) = p(\boldsymbol{\theta})$
- Iterate:
  - Remove the current estimate of $\tilde{f}_n(\boldsymbol{\theta})$ from q by division of Gaussians:

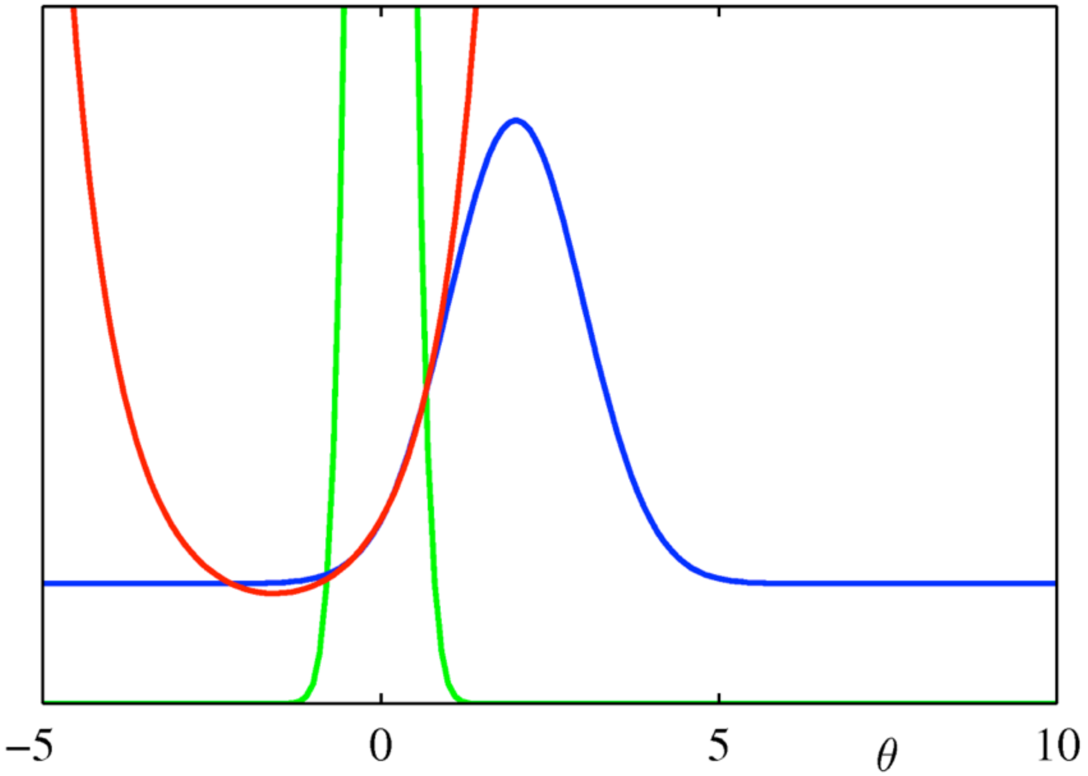
$$q_{-n}(\boldsymbol{\theta}) = \frac{q(\boldsymbol{\theta})}{\tilde{f}_n(\boldsymbol{\theta})} \qquad q_{-n}(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta} \mid \mathbf{m}_{-n}, v_{-n} I)$$

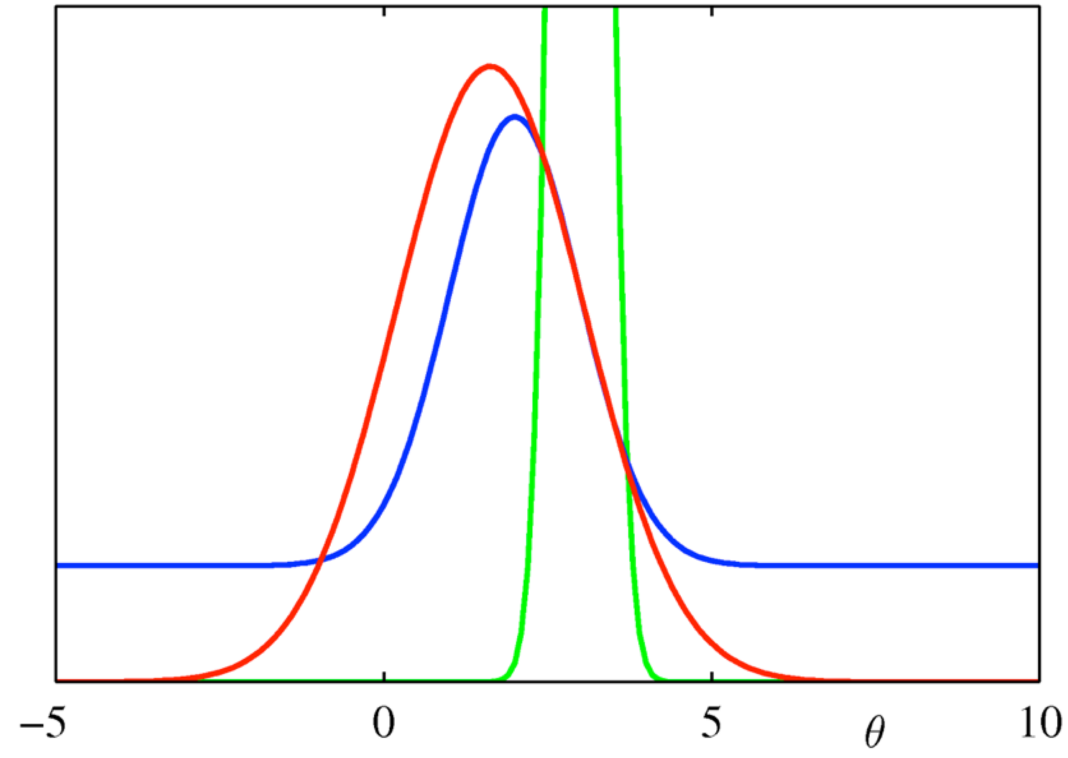
  - Compute the normalization constant:

$$Z_n = \int q_{-n}(\boldsymbol{\theta}) f_n(\boldsymbol{\theta}) d\boldsymbol{\theta}$$

  - Compute mean and variance of q^{new}
  - Update the factor $\tilde{f}_n(\boldsymbol{\theta}) = Z_n \frac{q^{\text{new}}(\boldsymbol{\theta})}{q_{-n}(\boldsymbol{\theta})}$

A 1D Example





- blue: true factor $f_n(\boldsymbol{\theta})$
- red: approximate factor $\tilde{f}_n(\boldsymbol{\theta})$
- green: cavity distribution $q_{-n}(\boldsymbol{\theta})$

The form of $q_{-n}(\boldsymbol{\theta})$ controls the range over which $\tilde{f}_n(\boldsymbol{\theta})$ will be a good approximation of $f_n(\boldsymbol{\theta})$

Summary

- Variational Inference uses approximation of functions so that the KL-divergence is minimal
- In mean-field theory, factors are optimized sequentially by taking the expectation over all other variables
- Variational inference for GMMs reduces the risk of overfitting; it is essentially an EM-like algorithm
- Expectation propagation minimizes the reverse KL-divergence of a single factor by moment matching; factors are in the exp. family

10. Sampling Methods

Sampling Methods

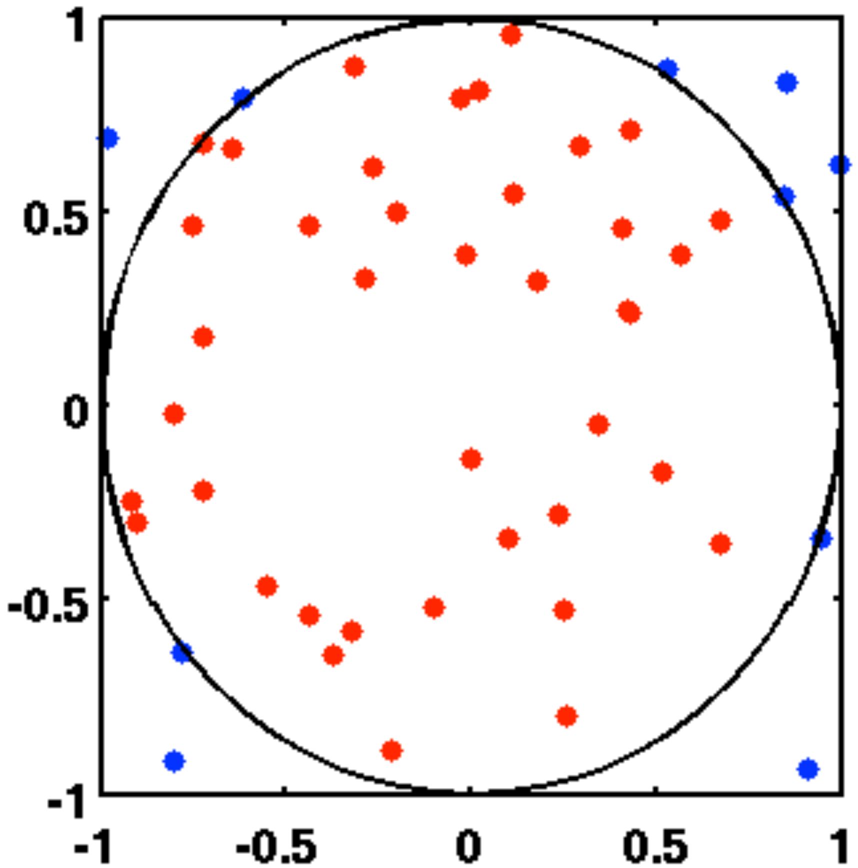
Sampling Methods are widely used in Computer Science

- as an **approximation** of a deterministic algorithm
- to represent **uncertainty** without a parametric model
- to obtain higher computational **efficiency** with a small approximation error

Sampling Methods are also often called **Monte Carlo Methods**

Example: Monte-Carlo Integration

- Sample in the bounding box
- Compute fraction of inliers
- Multiply fraction with box size



Non-Parametric Representation

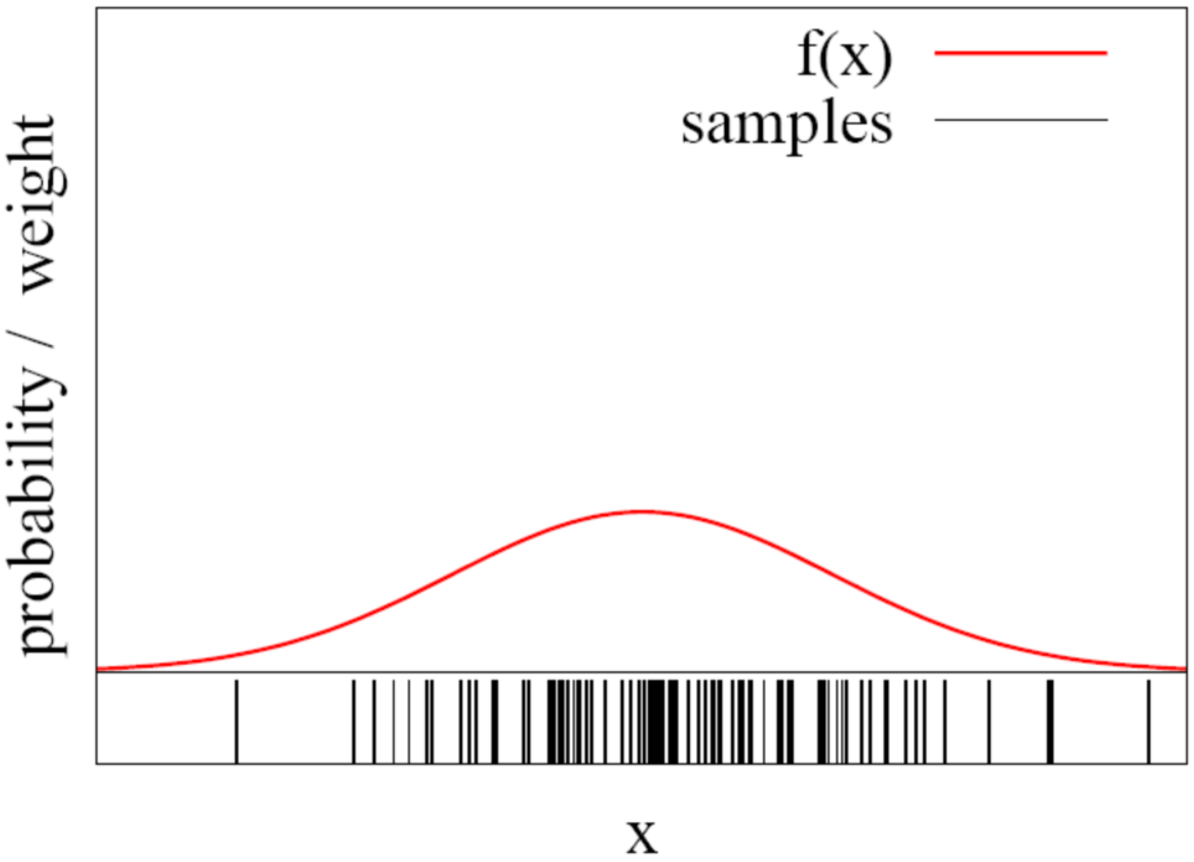
Probability distributions (e.g. a robot‘s belief) can be represeted:

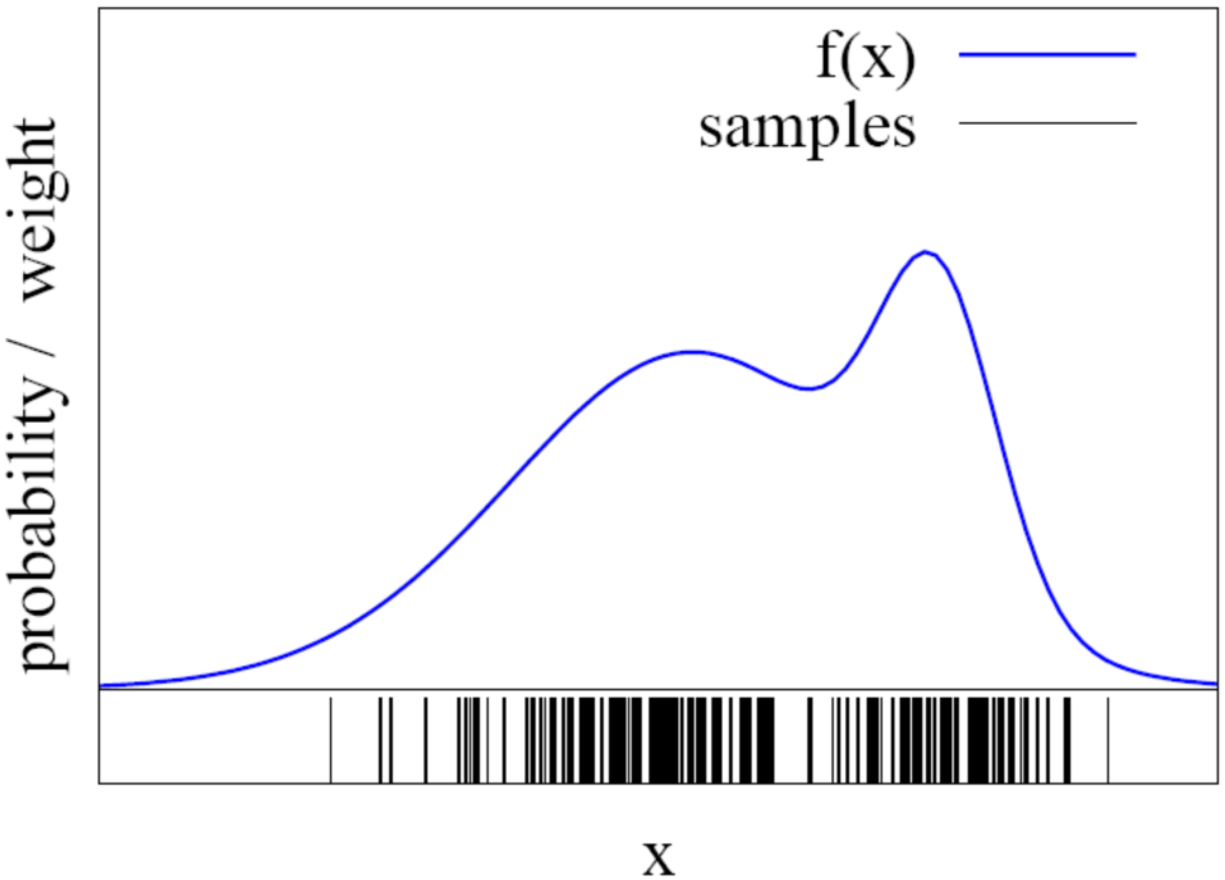
- **Parametrically**: e.g. using mean and covariance of a Gaussian
- **Non-parametrically:** using a set of *hypotheses* (samples) drawn from the distribution

Advantage of non-parametric representation:

- No restriction on the *type* of distribution (e.g. can be multi-modal, non- Gaussian, etc.)

Non-Parametric Representation





The more samples are in an interval, the higher the probability of that interval

But:

How to draw samples from a function/distribution?

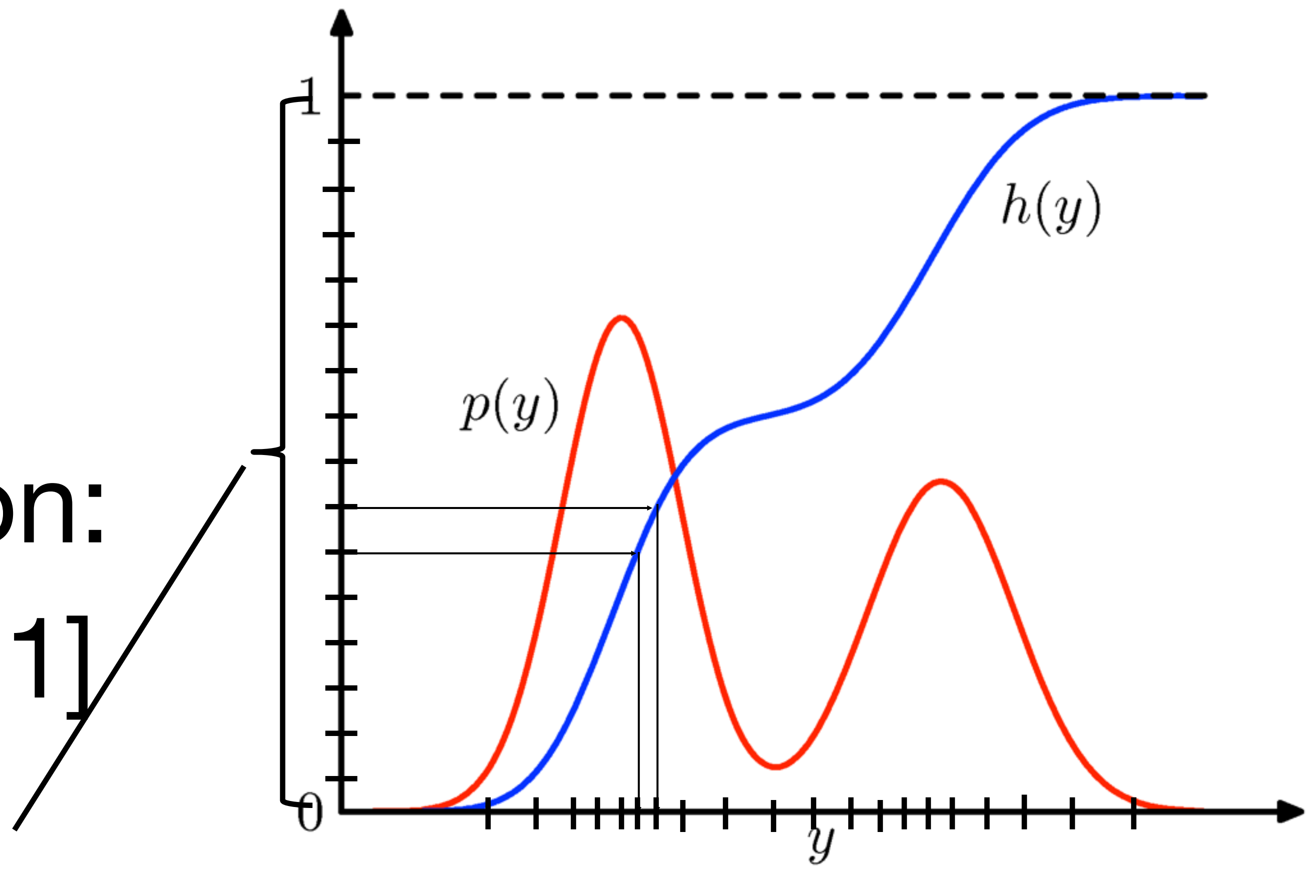
Sampling from a Distribution

There are several approaches:

- Probability transformation
  - Uses inverse of the c.d.f h
- Rejection Sampling
- Importance Sampling
- MCMC

$$h(y) = \int_{-\infty}^{y} p(\hat{y})d\hat{y}$$

"Cumulative distribution Function"



Probability transformation:

- Sample uniformly in [0,1]
- Transform using h^{-1}

But:

- Requires calculation of h and its inverse

Rejection Sampling

1. Simplification:

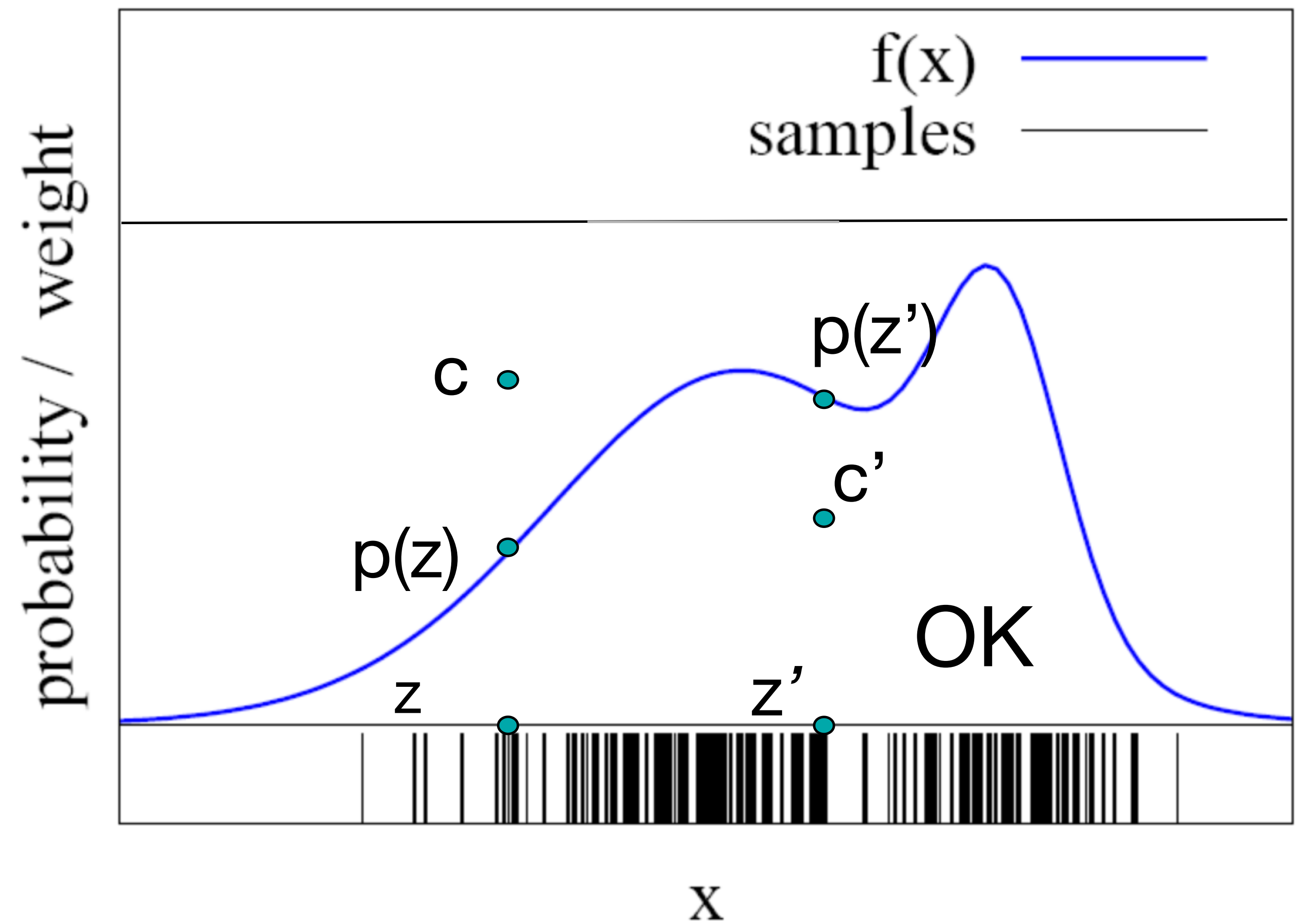
- Assume $p(z) < 1$ for all z
- Sample z uniformly
- Sample c from $[0, 1]$

- **If** $f(z) > c$:

  keep the sample

  otherwise:

  reject the sample

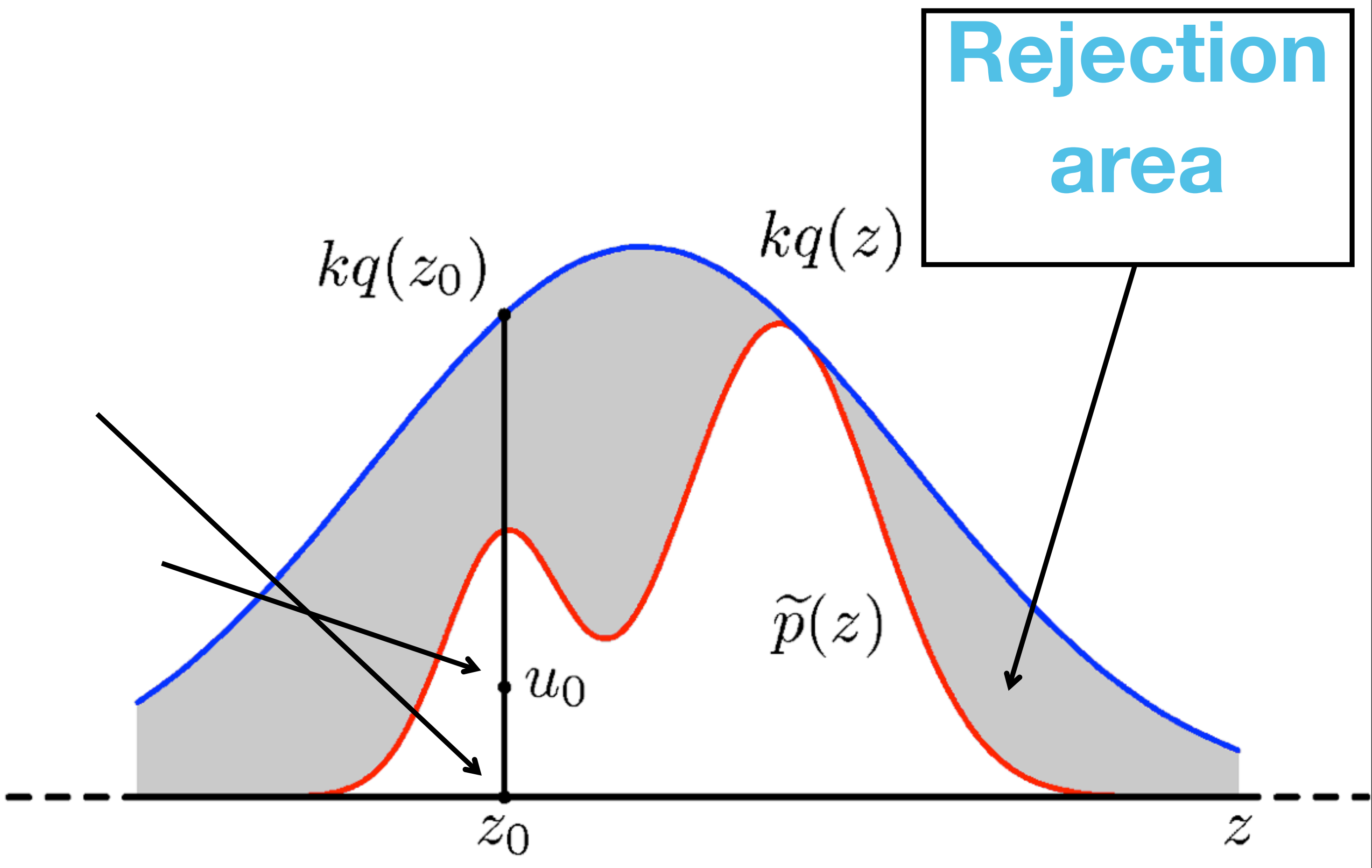


Rejection Sampling

2. General case:

Assume we can evaluate $p(z) = \frac{1}{Z_p}\tilde{p}(z)$ (unnormalized)

- Find **proposal distribution** q
  - Easy to sample from q
- Find k with $kq(z) \geq \tilde{p}(z)$
- Sample from q
- Sample uniformly from $[0, kq(z_0)]$
- Reject if $u_0 > \tilde{p}(z_0)$



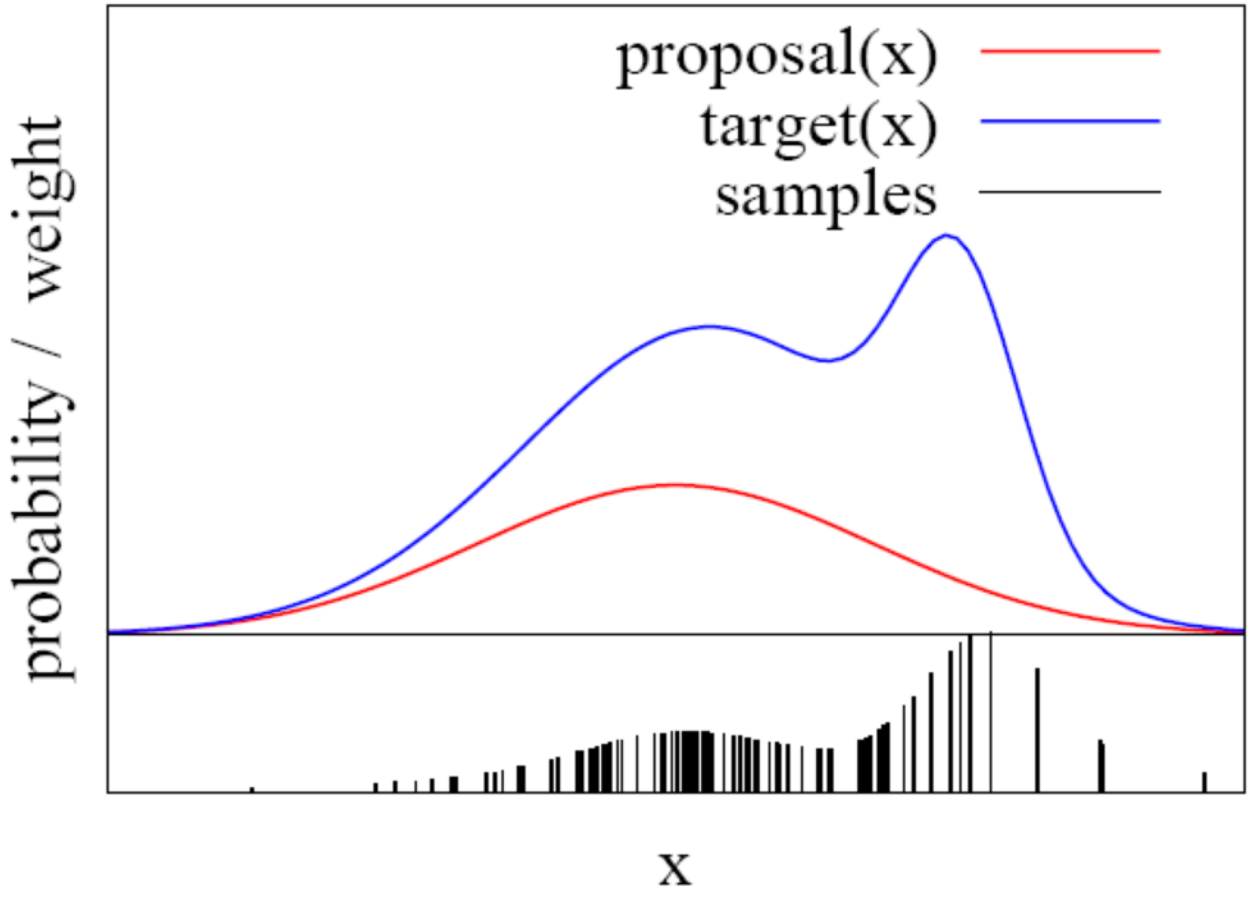
But: Rejection sampling is inefficient.

Importance Sampling

- **Idea:** assign an **importance weight** w to each sample
- With the importance weights, we can account for the "differences between p and q "

$$w(x) = p(x)/q(x)$$

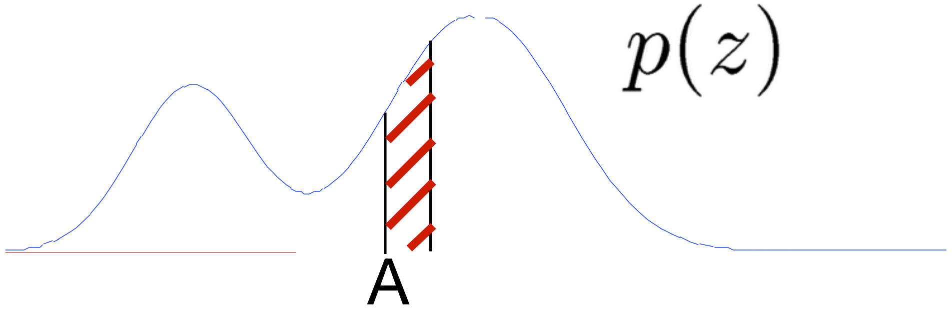
- p is called **target**
- q is called **proposal** (as before)



Importance Sampling

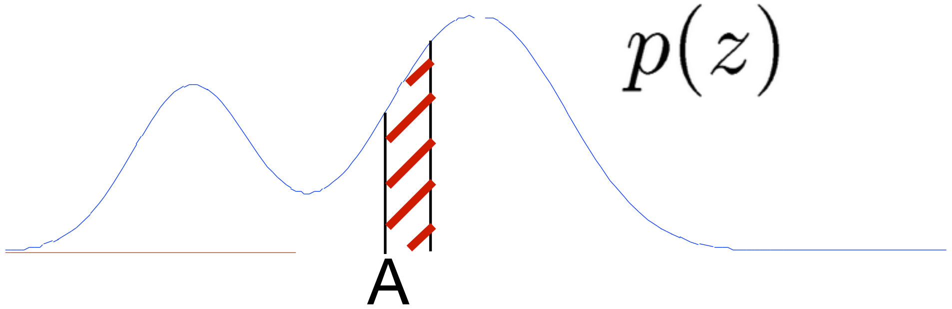
- **Explanation:** The prob. of falling in an interval A is the **area** under p
- This is equal to the expectation of the **indicator function** $I(x \in A)$

$$E_p[I(z \in A)] = \int p(z) I(z \in A) dz$$



Importance Sampling

- **Explanation:** The prob. of falling in an interval A is the **area** under p
- This is equal to the expectation of the **indicator function** $I(x \in A)$



$$E_p[I(z \in A)] = \int p(z) I(z \in A) dz$$

$$= \int \frac{p(z)}{q(z)} q(z) I(z \in A) dz = E_q[w(z) I(z \in A)]$$

Requirement: $p(x) > 0 \Rightarrow q(x) > 0$

Approximation with samples drawn from q: $E_q[w(z)I(z \in A)] \approx \frac{1}{L} \sum_{l=1}^{L} w(z_l) I(z_l \in A)$

The Particle Filter

- **Non-parametric** implementation of Bayes filter
- Represents the belief (posterior) $\text{Bel}(x_t)$ by a set of **random state samples.**
- This representation is **approximate.**
- Can represent distributions that are **not Gaussian**.
- Can model **non-linear** transformations.

Basic principle:

- Set of state hypotheses (“particles”)
- Survival-of-the-fittest

The Bayes Filter Algorithm (Rep.)

$$\text{Bel}(x_t) = \eta \, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1}) \text{Bel}(x_{t-1}) dx_{t-1}$$

Algorithm Bayes_filter $(\text{Bel}(x), d)$

1. if d is a sensor measurement z then
2. $\quad \eta = 0$
3. $\quad$ for all x do
4. $\quad\quad \text{Bel}'(x) \leftarrow p(z \mid x)\text{Bel}(x)$
5. $\quad\quad \eta \leftarrow \eta + \text{Bel}'(x)$
6. $\quad$ for all x do $\text{Bel}'(x) \leftarrow \eta^{-1}\text{Bel}'(x)$
7. else if d is an action u then
8. $\quad$ for all x do $\text{Bel}'(x) \leftarrow \int p(x \mid u, x')\text{Bel}(x')dx'$
9. return $\text{Bel}'(x)$

Mathematical Description

Set of weighted samples:

$$\mathcal{X}_t := \{\langle x_t^{[1]}, w_t^{[1]}\rangle, \langle x_t^{[2]}, w_t^{[2]}\rangle, \ldots, \langle x_t^{[M]}, w_t^{[M]}\rangle\}$$

State hypotheses

Importance weights

The samples represent the probability distribution:

$$p(x) = \sum_{i=1}^{M} w_t^{[i]} \cdot \delta_{x_t^{[i]}}(x)$$

Point mass distribution ("Dirac")

The Particle Filter Algorithm

Algorithm ***Particle_filter***$(\mathcal{X}_{t-1}, u_t, z_t)$ **:**

1. $\bar{\mathcal{X}}_t = \mathcal{X}_t = \emptyset$
2. **for** $m = 1$ **to** M **do**
3. sample $x_t^{[m]} \sim p(x_t \mid u_t, x_{t-1}^{[m]})$
4. $w_t^{[m]} \leftarrow p(z_t \mid x_t^{[m]})$
5. $\bar{\mathcal{X}}_t \leftarrow \bar{\mathcal{X}}_t \cup \langle x_t^{[m]}, w_t^{[m]} \rangle$
6. **for** $m = 1$ **to** M **do**

   draw i with prob. $\propto w_t^{[i]}$

   $\mathcal{X}_t \leftarrow \mathcal{X}_t \cup \langle x_t^{[i]}, 1/M \rangle$
7. **return** $\mathcal{X}_t$

Sample from proposal

Compute sample weights

Resampling

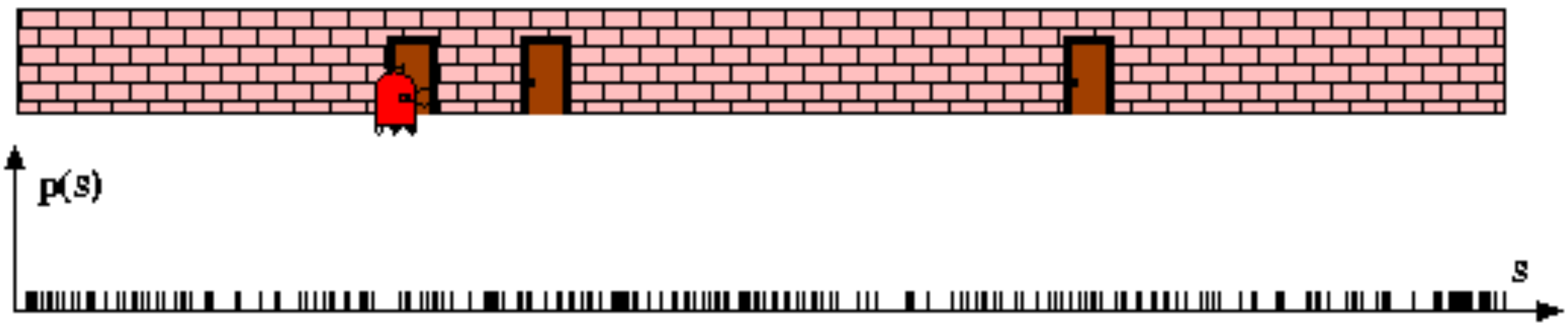
Localization with Particle Filters

- Each particle is a potential **pose** of the robot
- Proposal distribution is the motion model of the robot (**prediction step**)
- The observation model is used to compute the importance weight (**correction step**)

Randomized algorithms are usually called Monte Carlo algorithms, therefore we call this:

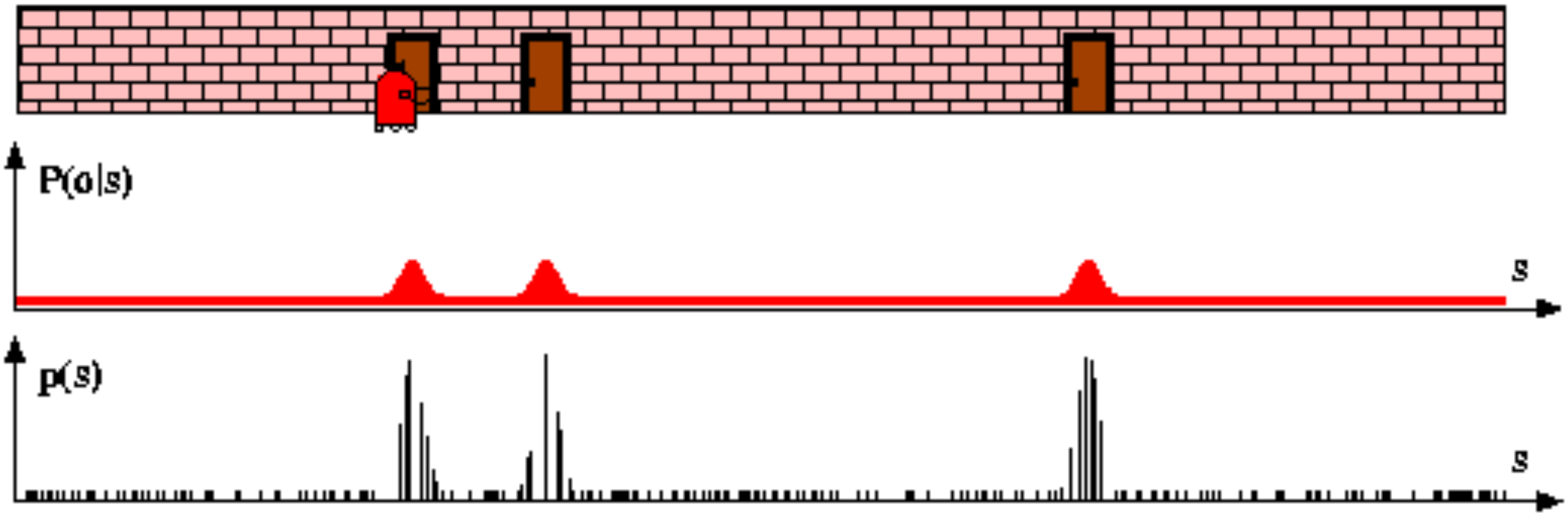
Monte-Carlo Localization

A Simple Example



- The initial belief is a uniform distribution (global localization).
- This is represented by an (approximately) uniform sampling of initial particles.

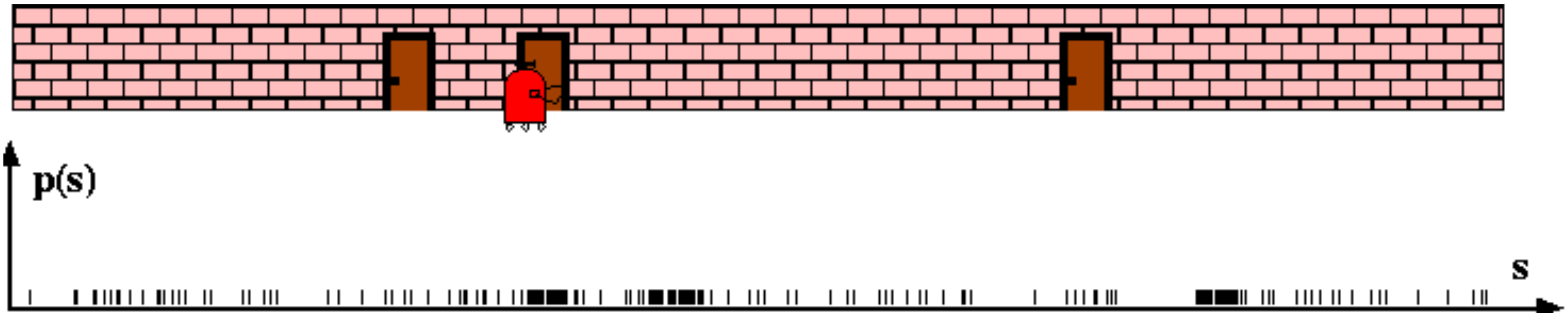
Sensor Information



The sensor model $p(z_t \mid x_t^{[m]})$ is used to compute the new importance weights:

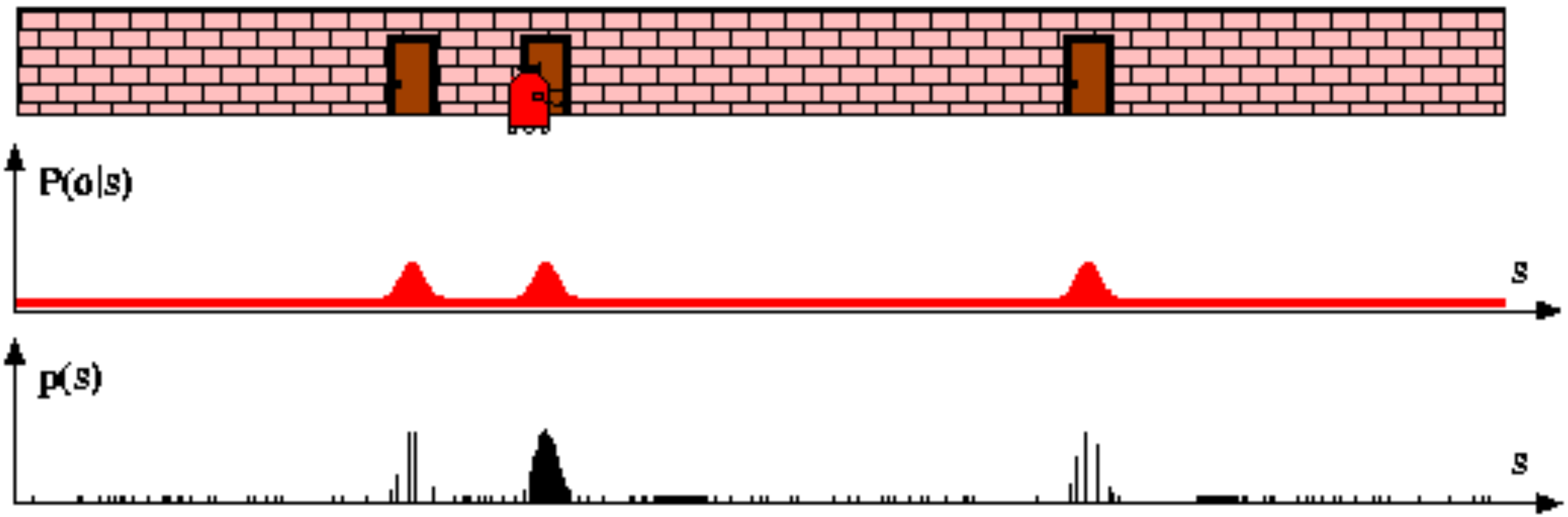
$$w_t^{[m]} \leftarrow p(z_t \mid x_t^{[m]})$$

Robot Motion



After resampling and applying the motion model $p(x_t \mid u_t, x_{t-1}^{[m]})$ the particles are distributed more densely at three locations.

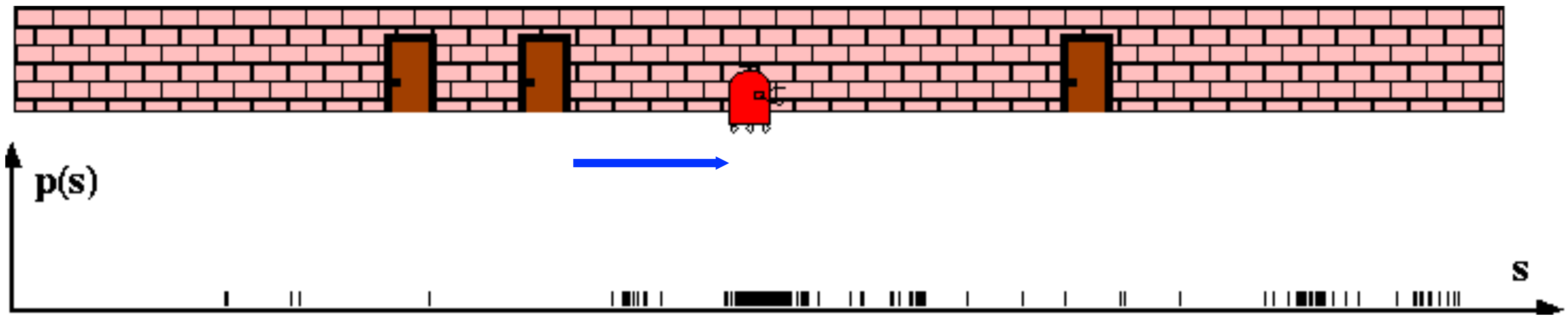
Sensor Information



Again, we set the new importance weights equal to the sensor model.

$$w_t^{[m]} \leftarrow p(z_t \mid x_t^{[m]})$$

Robot Motion



Resampling and application of the motion model:

One location of dense particles is left.

The robot is localized.

A Closer Look at the Algorithm…

Algorithm ***Particle_filter*** $(\mathcal{X}_t, u_t, z_t)$:

1. $\bar{\mathcal{X}}_t = \mathcal{X}_t = \emptyset$
2. **for** $m = 1$ **to** M **do**
3. sample $x_t^{[m]} \sim p(x_t \mid u_t, x_{t-1}^{[m]})$
4. $w_t^{[m]} \leftarrow p(z_t \mid x_t^{[m]})$
5. $\bar{\mathcal{X}}_t \leftarrow \bar{\mathcal{X}}_t + \langle x_t^{[m]}, w_t^{[m]} \rangle$
6. **for** $m = 1$ **to** M **do**
   draw i with prob. $\propto w_t^{[i]}$
   $\mathcal{X}_t \leftarrow \mathcal{X}_t \cup x_t^{[i]}$
7. **return** $\mathcal{X}_t$

Sample from proposal

Compute sample weights

Resampling

Sampling from Proposal

This can be done in the following ways:

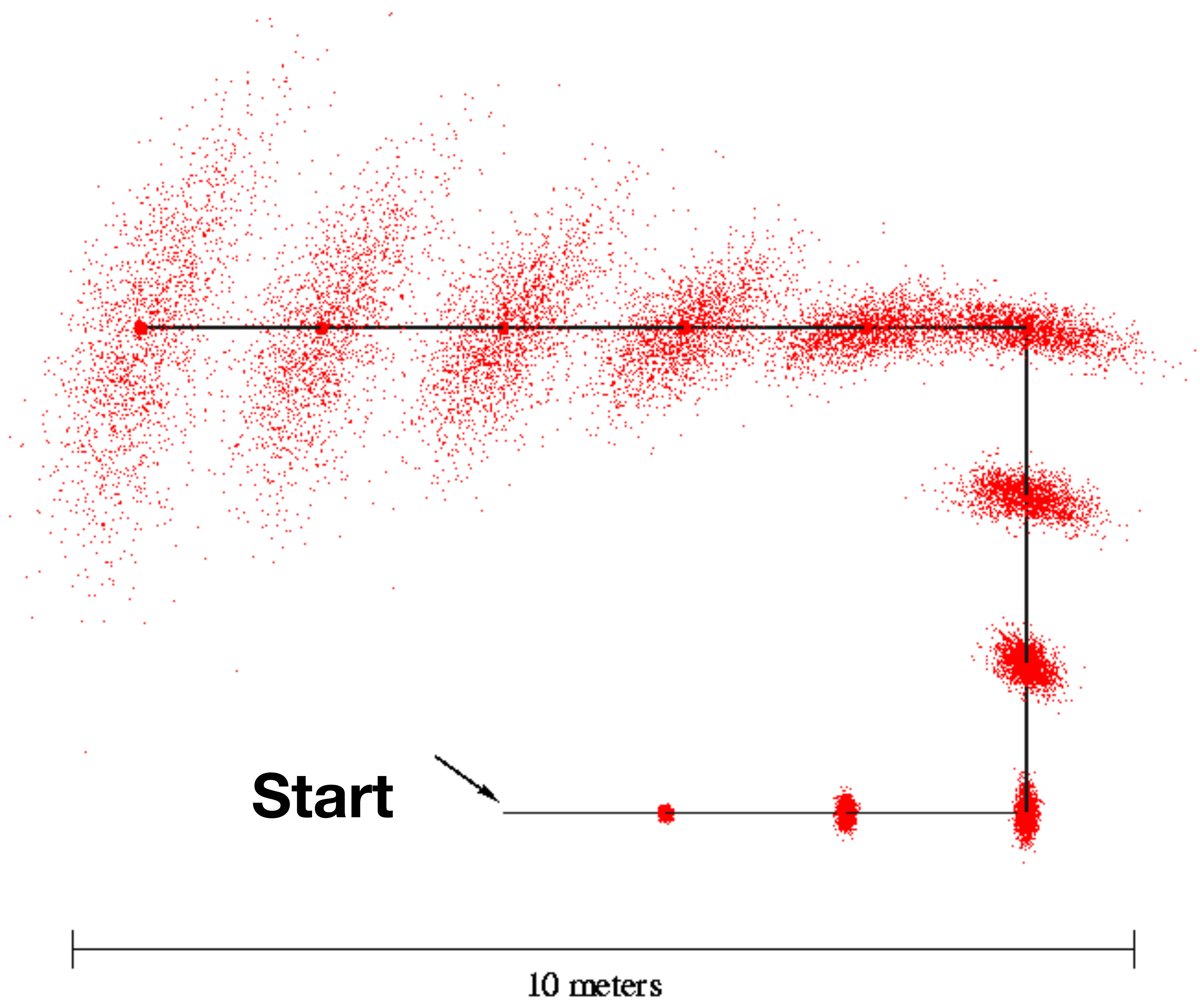
- Adding the motion vector to each particle directly (this assumes perfect motion) sample $x_t^{[m]} \sim p(x_t \mid u_t, x_{t-1}^{[m]})$
- Sampling from the motion model $p(x_t \mid u_t, x_{t-1}^{[m]})$, e.g. for a 2D motion with translation velocity v and rotation velocity w we have:

$$\mathbf{u}_t = \begin{pmatrix} v_t \\ w_t \end{pmatrix} \qquad \mathbf{x}_t = \begin{pmatrix} x_t \\ y_t \\ \theta_t \end{pmatrix}$$

Position: x_t, y_t

Orientation: θ_t

Motion Model Sampling (Example)



Computation of Importance Weights

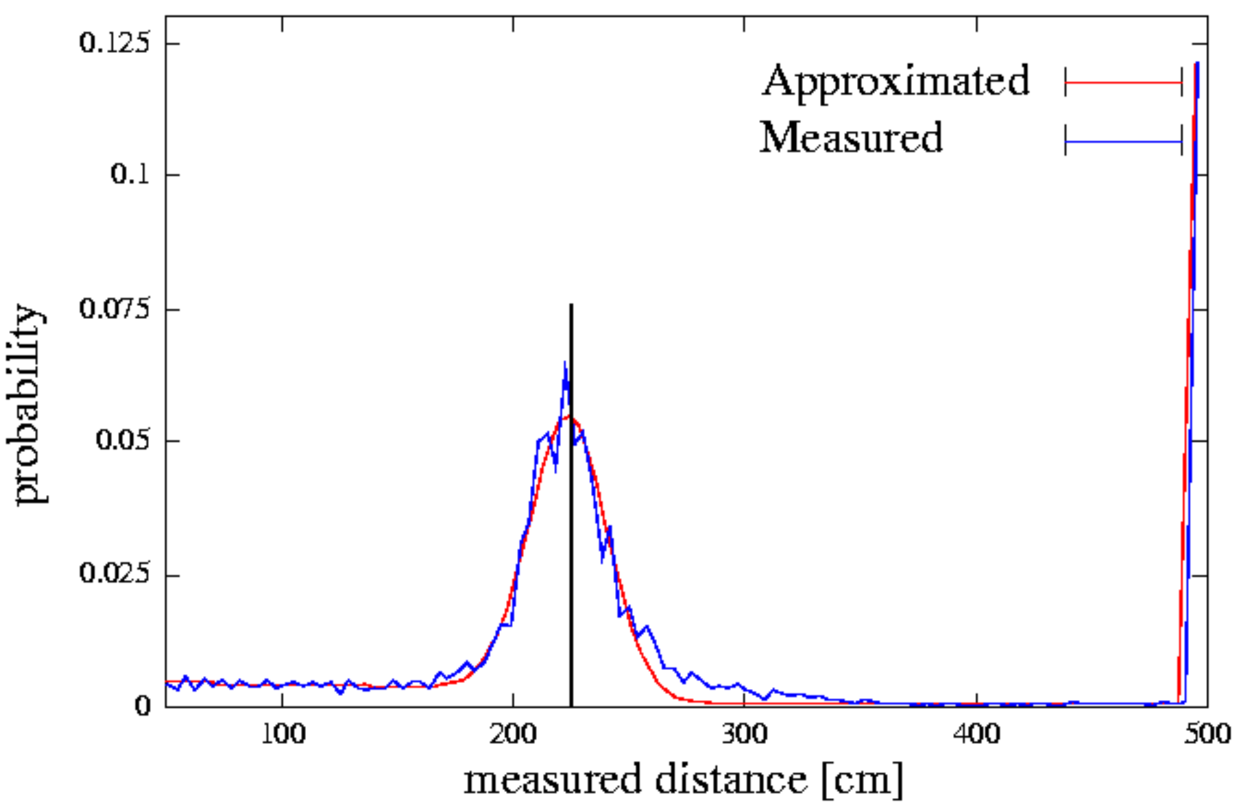
Computation of the sample weights:

- Proposal distribution: $g(x_t^{[m]}) = p(x_t^{[m]} \mid u_t, x_{t-1}^{[m]})\mathrm{Bel}(x_{t-1}^{[m]})$
  (we sample from that using the motion model)
- Target distribution (new belief): $f(x_t^{[m]}) = \mathrm{Bel}(x_t^{[m]})$
  (we can not directly sample from that → importance sampling)
  $w_t^{[m]} \leftarrow p(z_t \mid x_t^{[m]})$
- Computation of importance weights:

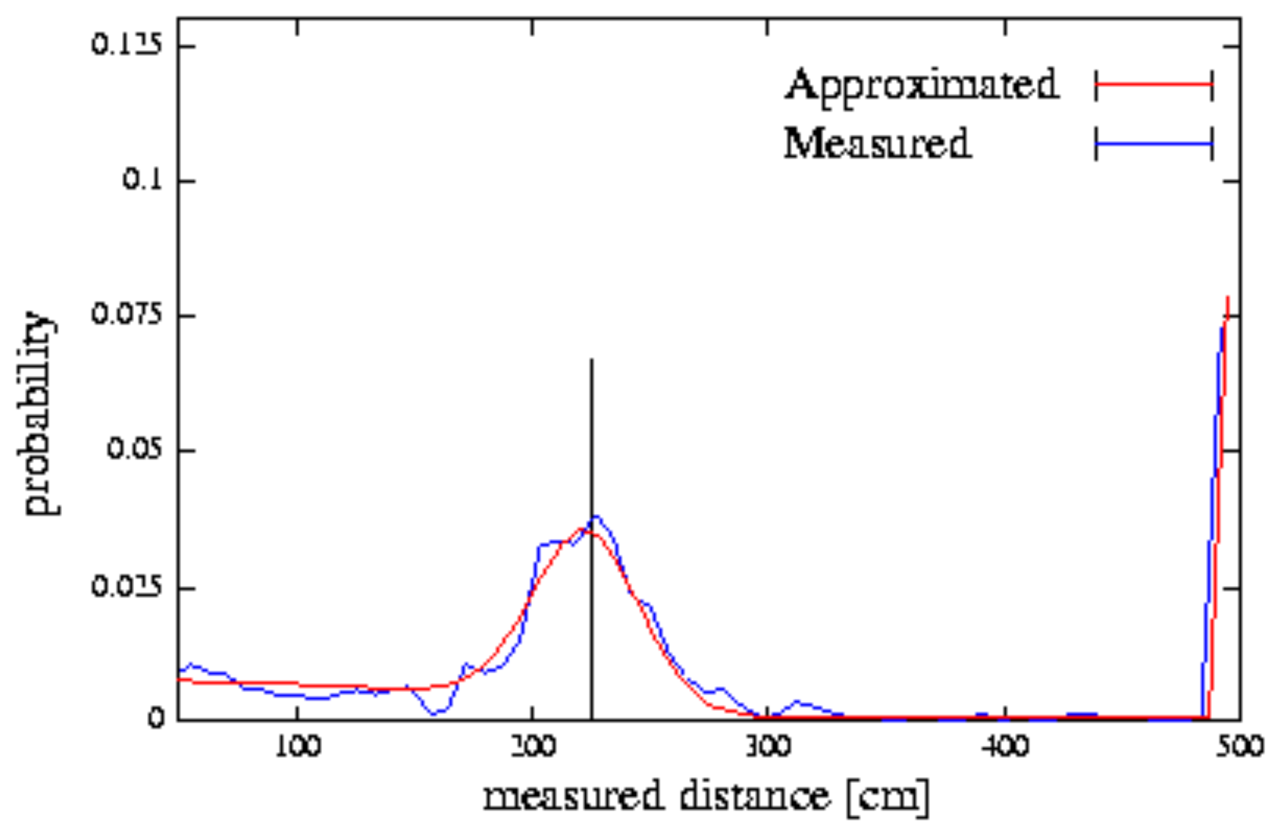
$$w_t^{[m]} = \frac{f(x_t^{[m]})}{g(x_t^{[m]})} \propto \frac{p(z_t \mid x_t^{[m]})p(x_t^{[m]} \mid u_t, x_{t-1}^{[m]})\mathrm{Bel}(x_{t-1}^{[m]})}{p(x_t^{[m]} \mid u_t, x_{t-1}^{[m]})\mathrm{Bel}(x_{t-1}^{[m]})} = p(z_t \mid x_t^{[m]})$$

Proximity Sensor Models

- How can we obtain the sensor model $p(z_t \mid x_t^{[m]})$?
- Sensor Calibration:



Laser sensor



Sonar sensor

Resampling

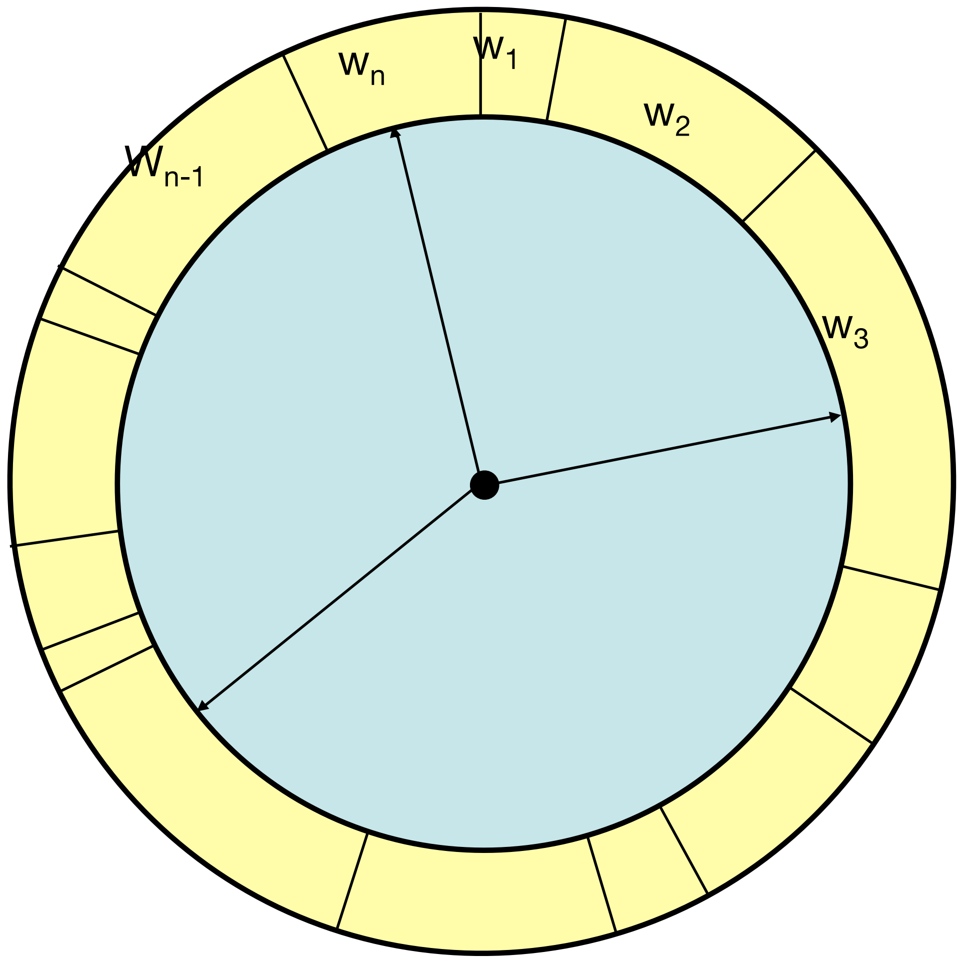
- Given: Set $\bar{\mathcal{X}}_t$ of weighted samples.
- Wanted : Random sample, where the probability of drawing x_i is equal to w_i.
- Typically done M times with replacement to generate new sample set $\mathcal{X}_t$

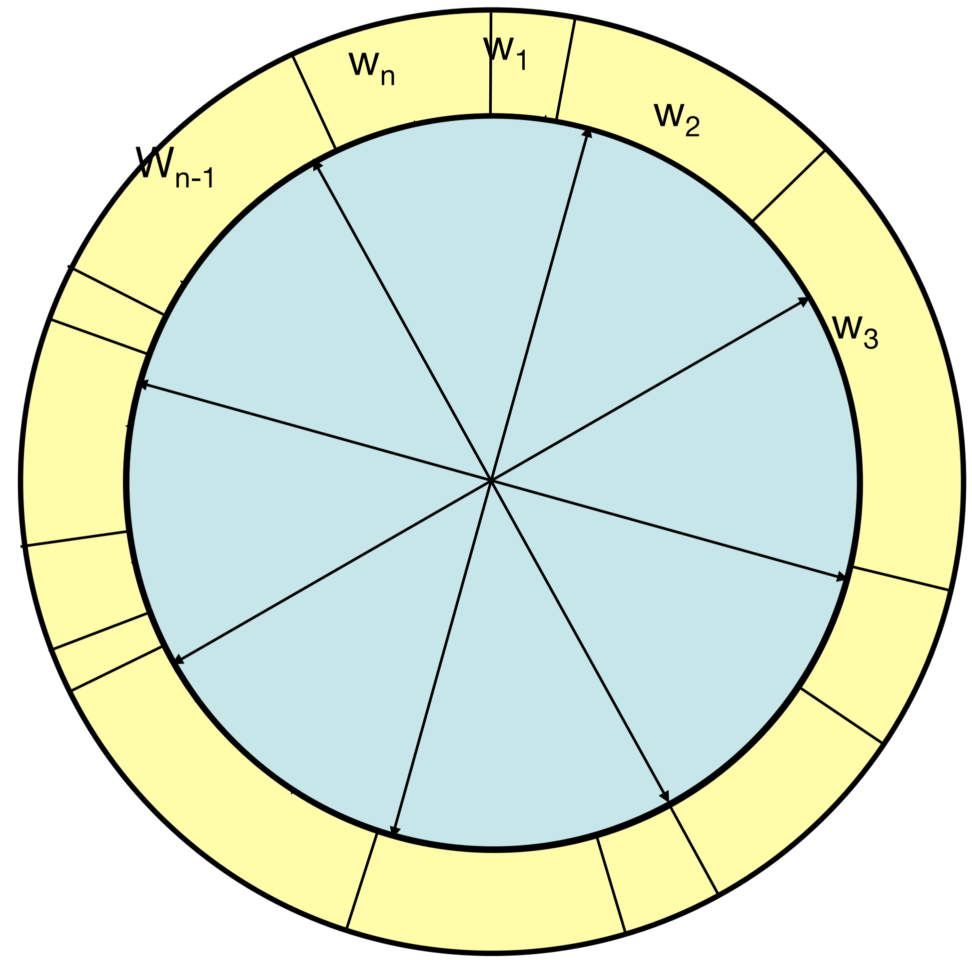
for $m = 1$ **to** M **do**

draw i with prob. $\propto w_t^{[i]}$

$\mathcal{X}_t \leftarrow \mathcal{X}_t \cup x_t^{[i]}$

Resampling

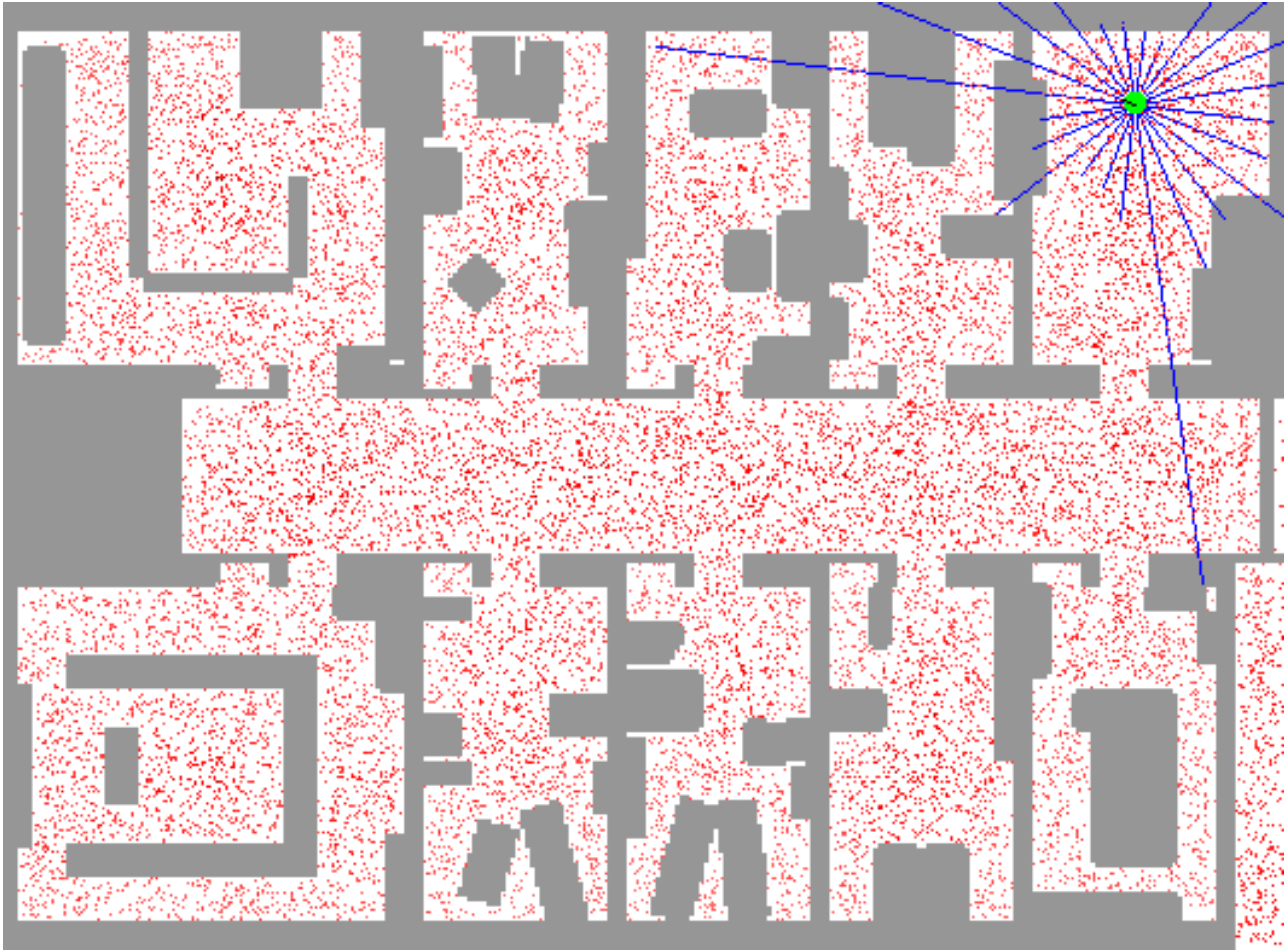




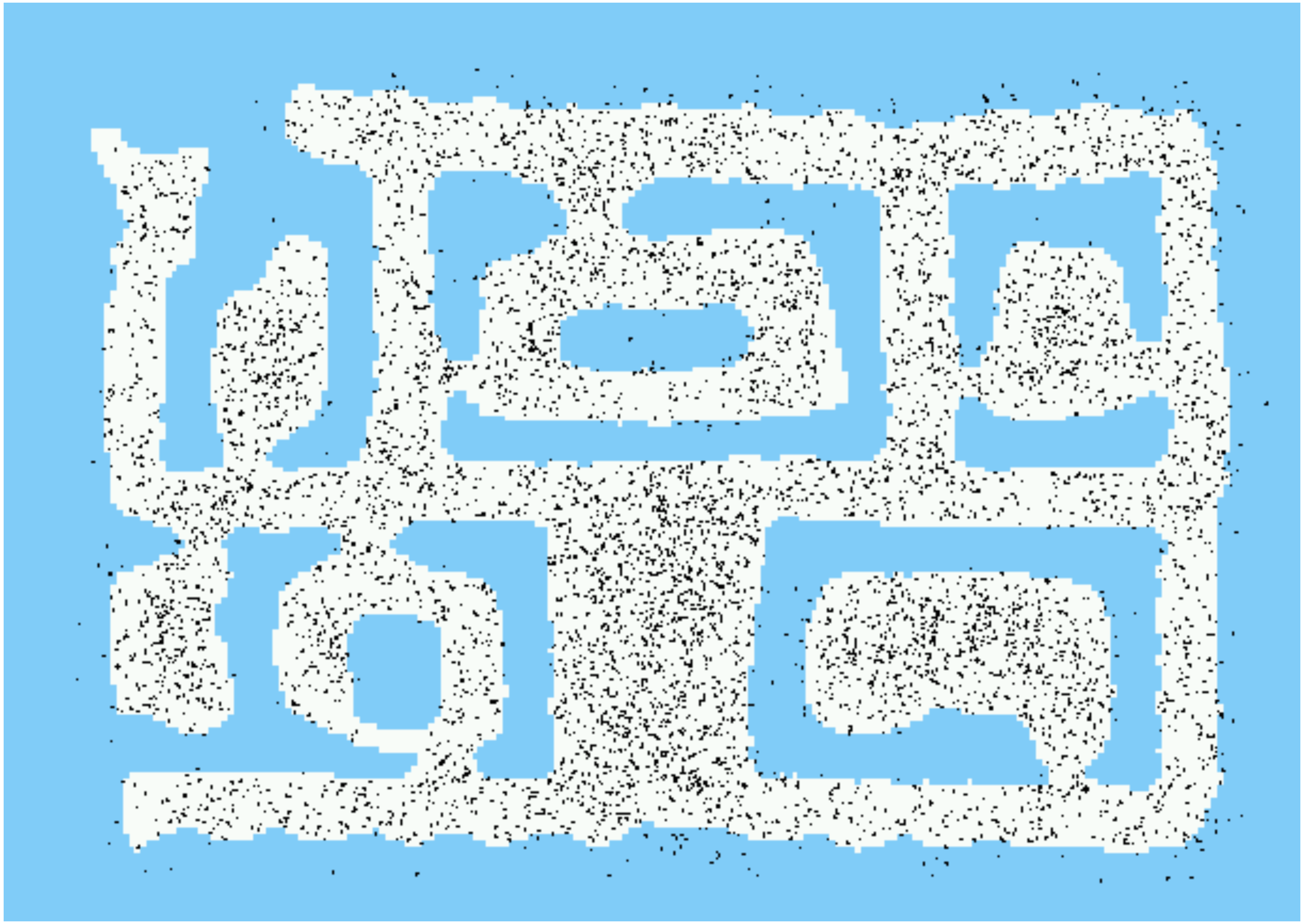
- Standard n-times sampling results in high variance
- This requires more particles
- O(nlog n) complexity

- Instead: low variance sampling only samples once
- Linear time complexity
- Easy to implement

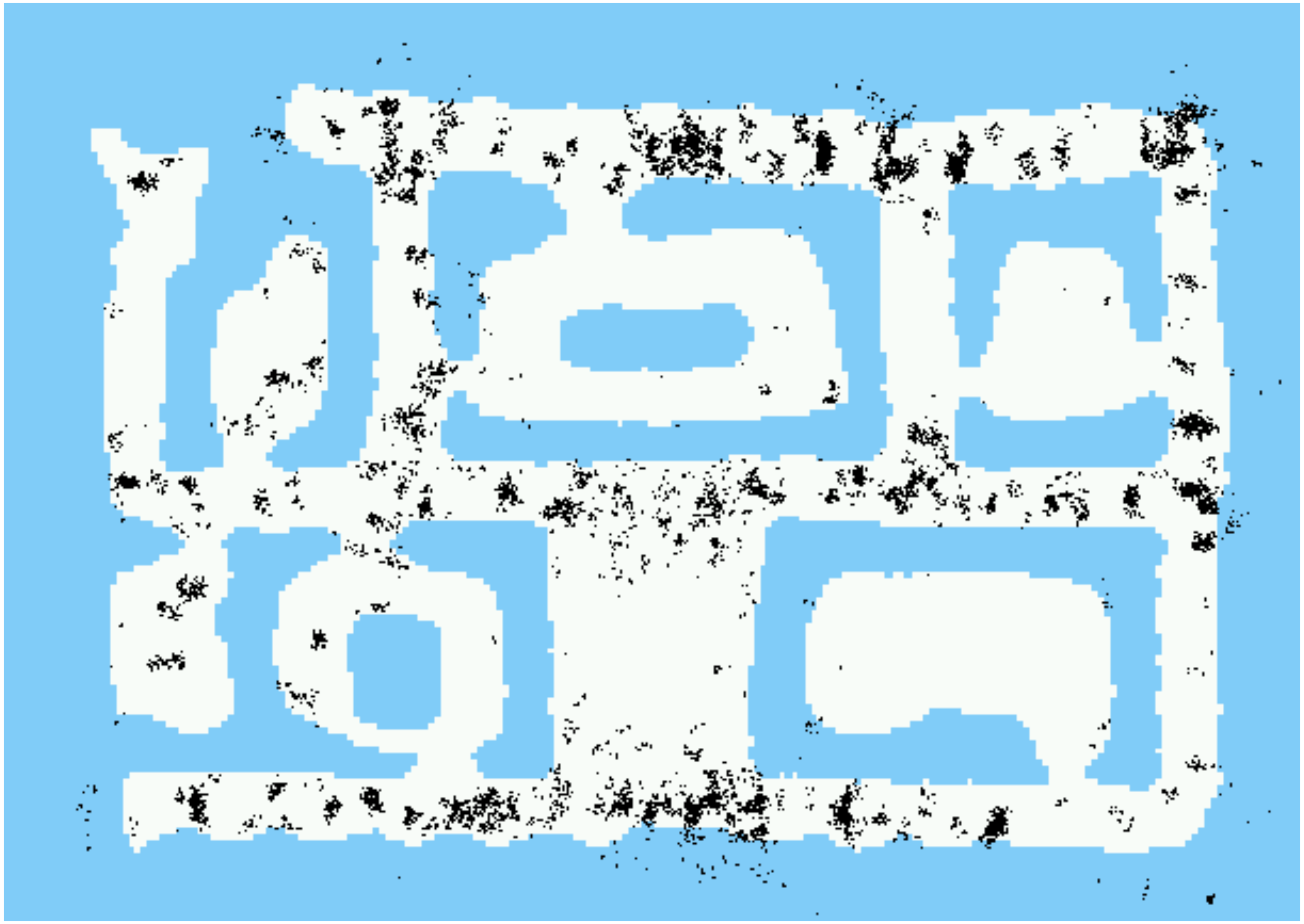
Sample-based Localization (sonar)



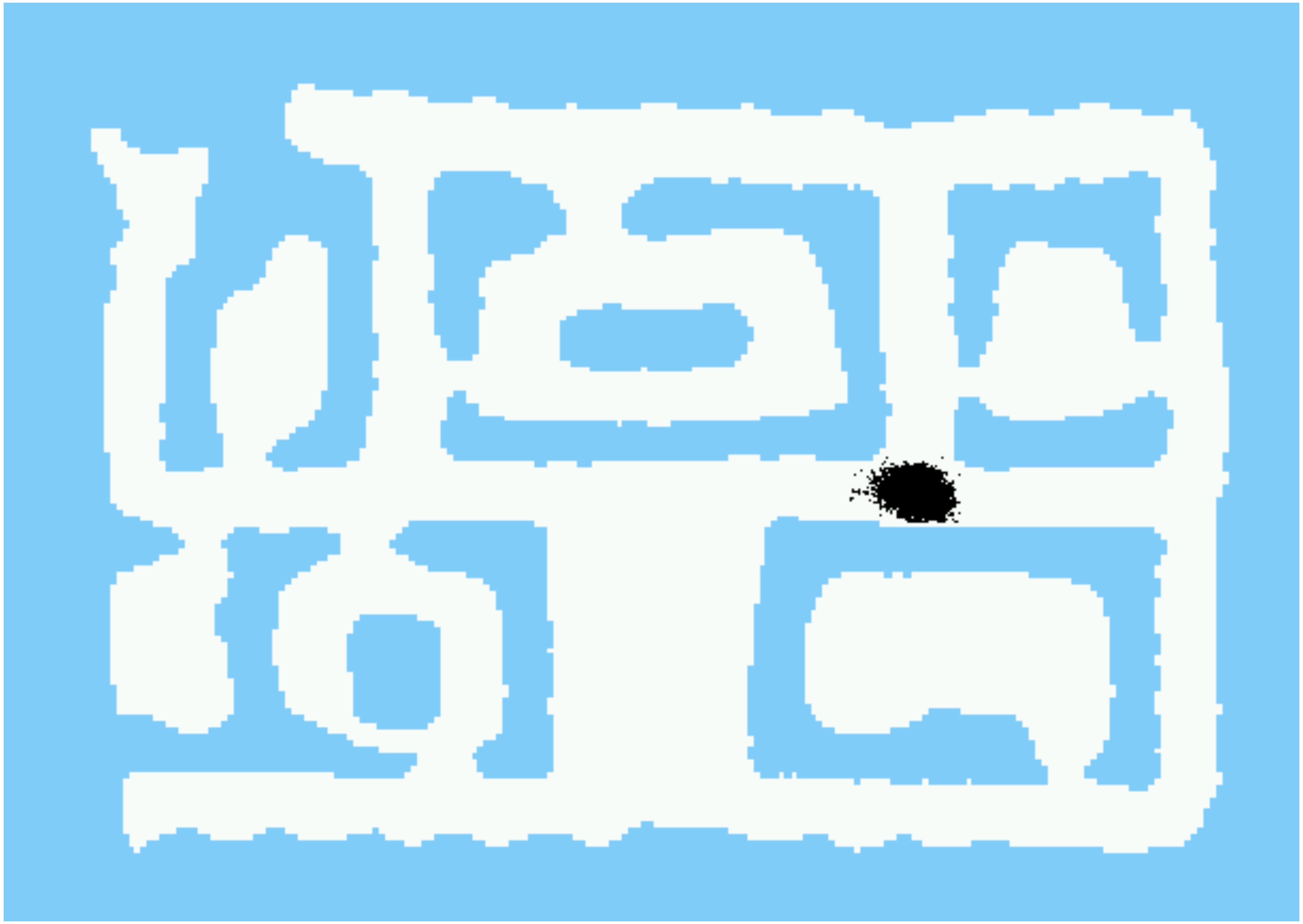
Initial Distribution



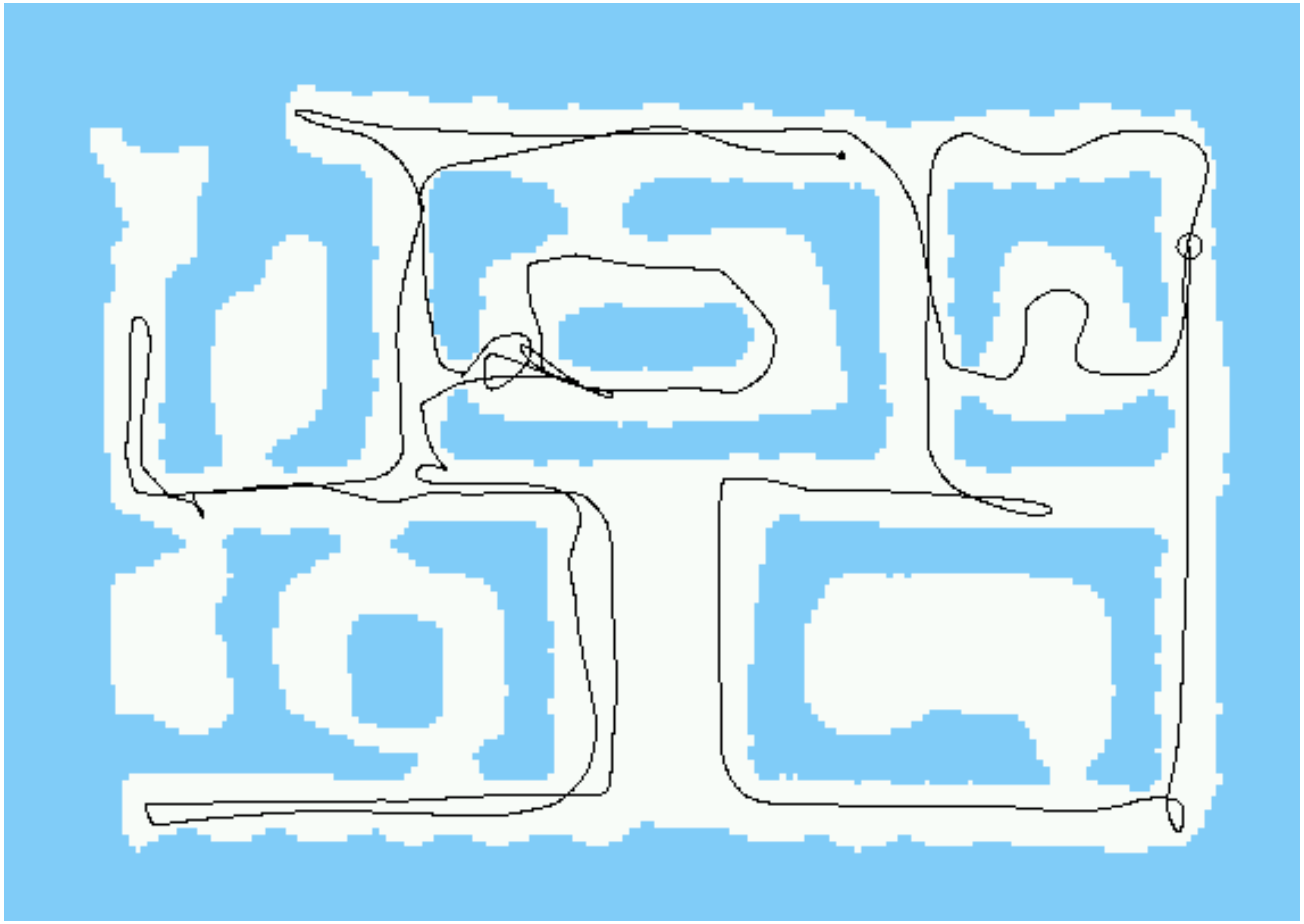
After Ten Ultrasound Scans



After 65 Ultrasound Scans



Estimated Path



Kidnapped Robot Problem

The approach described so far is able to

- track the pose of a mobile robot and to
- globally localize the robot.

- How can we deal with localization errors (i.e., the kidnapped robot problem)?

Idea: Introduce uniform samples at every resampling step

- This adds new hypotheses and reduces the

Summary

- There are mainly 4 different types of sampling methods: Transformation method, rejections sampling, importance sampling and MCMC
- Transformation only rarely applicable
- Rejection sampling is often very inefficient
- Importance sampling is used in the particle filter which can be used for robot localization
- An efficient implementation of the resampling step is the low variance sampling