Variational Methods for Computer Vision: Exercise Sheet 2

Exercise: 07 November 2013

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. A multidimensional filter ist called separable, if it can be decomposed in one dimensional filter operations. Prove that the convolution of an image f with a Gaussian kernel K of standart deviation $\sigma > 0$:
$$K(x,y) := \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right).$$
can be written as the convolution with two one-dimensional filters:
$$k_1(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma}\right). \quad \text{and} \quad k_2(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right).$$
Hence :
$$f(x,y) * K(x,y) = (f(x,y) * k_1(x)) * k_2(y)$$
Explain why the separability of a filter is a desirable property.

2. Let $f \in C^2(\Omega; \mathbb{R})$ with $\Omega \subset \mathbb{R}^2$ be a real valued function and let $R \in SO(2)$ be a rotation matrix. Let $\tilde{f}(x) := f(R \cdot x)$ be a rotated version of f. Prove that the magnitude of the gradient and the Laplace operator are rotationally covariant by showing the following identities:
   (a) $R\nabla\tilde{f}(x) = (\nabla f)(R \cdot x)$
   (b) $\|\nabla\tilde{f}(x)\| = \|(\nabla f)(R \cdot x)\|$
   (c) $\Delta\tilde{f}(x) = (\Delta f)(R \cdot x)$

   Reminder: $R \in \mathrm{SO}(2)$ denotes 2×2 matrices with $\det(R) = 1$ and $R^T R = \mathbb{1}$. and can be written as:
$$R = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$
   such that $\alpha \in [0, 2\pi]$.

3. The general diffusion equation for a function can be written as follows:
$$\partial_t u(x,t) = \mathrm{div}(g(u,x)\nabla u(x,t))$$
Where $u \in C^2(\Omega \times \mathbb{R}_0^+; \mathbb{R})$ with $\Omega \subset \mathbb{R}^2$ describes the complete diffusion process and solves the partial differential equation. Prove the following identities:
   (a) *linear homogeneous diffusion*:
$$\mathrm{div}(g\nabla u(x)) = g\Delta u(x) \quad s.t. \quad g \in \mathbb{R}$$
   (b) *linear inhomogeneous diffusion*:
$$\mathrm{div}(g(x)\nabla u(x)) = g(x)\Delta u(x) + \langle \nabla g(x), \nabla u(x) \rangle \quad s.t. \quad g \in C^1(\Omega; \mathbb{R})$$

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. In the last exercise we implemented an image convolution with the Gaussian kernel using the straightforward definitions. Considering the results from the first theoretical exercise, write a Matlab script that implements a separable convolution with a Gaussian kernel.

2. Download the archive file `vmcv_ex01.zip` from the homepage and unzip it in you home folder. Use the template file `difusion_filter.m` for a non linear diffusion filter and complete the missing code at line 58. Test the script on the image `lena.png`.

3. Create a video using the `avifile` command and compare the results.

Matlab-Tutorials:
`http://www.math.utah.edu/lab/ms/matlab/matlab.html`
`http://www.math.ufl.edu/help/matlab-tutorial/`
`http://www.glue.umd.edu/~nsw/ench250/matlab.htm`