Variational Methods for Computer Vision: Exercise Sheet 4

Exercise: 21 November 2013

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. Let $u \in C^2(\Omega; \mathbb{R})$ be a real valued function and $\Omega \subset \mathbb{R}$. And let

$$E(u) = \int_\Omega \mathcal{L}(u(x), u', u'') \, \mathrm{dx}$$

   be a real valued Gâteaux differentiable functional which depends on:

$$u(x) \quad , \quad u' = \frac{\mathrm{d}}{\mathrm{dx}} u(x) \quad , \quad \text{and} \quad u'' = \frac{\mathrm{d}^2}{\mathrm{dx}^2} u(x) \quad .$$

   Calculate the Gâteaux derivative of $E(u)$:

$$\left. \frac{\mathrm{d}E(u)}{\mathrm{du}} \right|_h$$

   for any differentiable direction h.

2. Let $u \in C^1(\mathbb{R}^3; \mathbb{R})$ be a real valued function. And let

$$E(u) = \int_\Omega \mathcal{L}(u(x), \nabla u(x)) \, \mathrm{dx}$$

   be a real valued Gâteaux differentiable functional. Calculate the Gâteaux derivative of $E(u)$.

3. Let $u \in C^1(\Omega; \mathbb{R})$ be a real valued function with $\Omega \subset \mathbb{R}^2$. Derive the Euler-Lagrange Equation of the following energies:

   (a)
$$E_1(u) = \int_\Omega |\nabla u(x)| \, \mathrm{dx}$$

   (b)
$$E_2(u) = \int_\Omega \sqrt{\nabla u(x)^T D(x) \nabla u(x)} \, \mathrm{dx}$$

   where $D(x) \in \mathbb{R}^{2\times 2}$ is a 2×2 real valued Matrix. The first energy is the so called Total Variation of function u.

Part II: Practical Exercises

This exercise is to be solved **during the tutorial**.

1. The Rudin Osher Fatemi Functional is one of the first models for image denoising based on Energy minimization. The ROF model pocesses the nice property of removing noise in the image while preserving discontinuities. The ROF energy functional can be formulated as follows:

$$E_{ROF}(u) := \int_{\Omega} |\nabla u(x)| \, \mathrm{dx} + \frac{1}{2\lambda} \int_{\Omega} (u - f)^2 \, \mathrm{dx}$$

   Where $\Omega \subset \mathbb{R}^2$ represents the image domain, $u : \Omega \to \mathbb{R}$ denotes the optimization variable and $f : \Omega \to \mathbb{R}$ stands for the input image. Hence the ROF model is a weighted sum of the Total Variation energy and a data similarity measure also called data term. The parameter λ can be tuned in order to change the amount of smoothness.

   (a) In the theoretical exercises we calculated the Euler-Lagrange equation of the Total Variation of a function u which is a part of the ROF model. Write down the complete optimality condition for the ROF model.

   (b) Obtain an optimal $u^* = \mathrm{argmin}_u(E_{ROF})$ by applying the gradient descent scheme shown in the lecture. Explain the edge preserving properties by looking at the gradient descent equation of E_{ROF}.

Matlab-Tutorials:
http://www.math.utah.edu/lab/ms/matlab/matlab.html
http://www.math.ufl.edu/help/matlab-tutorial/
http://www.glue.umd.edu/˜nsw/ench250/matlab.htm