Variational Methods for Computer Vision: Exercise Sheet 6

Exercise: December 5th, 2013

Part I: Theory

The following exercises should be **solved at home**. You do not have to hand in your solutions, however, writing it down will help you present your answer during the tutorials.

1. Let $\Omega = [-5; 5] \times [-5; 5]$ be a rectangular area and let $I : \Omega \to [0; 1]$ be an image given by :

$$I(x, y) = \begin{cases} 1 & \text{if} \quad x^2 + y^2 \leq 1 \\ 0 & \text{else} \end{cases}$$

   Furthermore, let $C : [0, 1] \to \Omega$ be a curve represented by a circle centered at the origin having radius r and local curvature κ_C.

   (a) Write down the two-region piecewise constant Mumford-Shah Functional with length regularity weighted by ν.

   (b) Write down the corresponding gradient descent for the two cases $r > 1$ and $r \leq 1$.

   (c) Show that the Gâteaux-Derivative at $r = 1$ is not continuous.

   (d) Why is $\nu \leq 1$ a good choice in order to obtain good segmentation results? What is the ideal choice for ν in our example?

2. Let $Q := [0, 1] \times [0, 1]$ be a rectangular area and let $\vec{v} = (a(x, y), b(x, y)) \in \mathbb{R}^2$ be a differentiable vector field defined on Q.

   (a) Prove Green's theorem:

$$\iint_Q b_x(x, y) - a_y(x, y)\; dx\; dy = \oint_{\partial Q} (a\; dx + b\; dy)$$

   Assume the boundary curve ∂Q to be oriented counter-clockwise.
   Hint: Use the fundamental theorem of calculus.

   (b) Let $Q \subset \mathbb{R}^2$ be an area that can be represented as a disjoint union of a finite number of squares $Q_1, ..., Q_n$.
   Why does Green's theorem also hold for this set Q?

Part II: Practical Exercises

This exercise is to be solved during the tutorial.

Super-Resolution from Video

In the lecture we encountered the concept of super-resolution from video. The key idea of super resolution is to exploit redundancy available in multiple frames of a video.
Assuming that each input frame is a blurred and down-sampled version of a higher resolved image u the high-resolution image can be recovered as the minimum of the following energy functional:

$$E(u) = \sum_{i=1}^{n} \int_{\Omega} (ABS_i\, u(x) - U f_i)^2 \, dx + \lambda \int_{\Omega} |\nabla u(x)| \, dx, \tag{1}$$

where:

- n is the number of frames in the video.
- The Linear Operator B denotes a Gaussian Blurring.
- The up-sampling operator U simply replaces every pixel with four pixels of the same color.
- In order to be able to compare image u with the up-sampled version of f_i which is constant block-wise we apply the linear averaging operator A on u which assigns every block of pixels the mean values of the pixels in that block.
- The linear operator S_i accounts for the coordinate shift by motion s_i hence: $S_i u = u(x + s_i(x))$

1. In the following we construct a super-resolution problem:

   (a) Download the file `scene.jpg`.

   (b) Convolve the input image with a Gaussian Kernel to simulate blurring.

   (c) Create 6 versions of the blurred image shifted in x direction by exactly one pixel, hence:

   $$f_i = f(x + i, y), \quad \text{for } i = 1, .., 6.$$

   In order to account for the boundary consider taking cropped images from the interior of the original image.

   (d) Down-sample the images f_i by factor 2 by using the `imresize` function in Matlab with nearest-neighbor interpolation.

2. In what follows we are going to minimize the above functional in order to obtain a super-resolved image from our input images f_i.

   (a) Derive the Euler-Lagrange Equation of E and the corresponding gradient descent scheme.

   (b) Compute the matrix representations of the linear operators A, B, S and U. The goal is to have a matrix-vector-multiplication, where u and f are stacked in a vector using the Matlab-command `reshape`.
   Hint: Use sparse data structures `spdiags,speye` and the kronecker product `kron`.

   (c) Compute $\text{argmin}_u\, E(u)$ by means of the gradient descent derived in 2(a).